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Coalgebraic Foundations of the Method of Divided Differences

PHILIP S. HIRSCHHORN

Department of Mathematics, Wellesley College, Wellesley, Massachusetts 02181

AND

LOUISE ARAKELIAN RAPHAEL*

Department of Mathematics, Howard University, Washington, D.C. 20059

1. INTRODUCTION

1.1. *Introduction*

The analogy between properties of the function e^x and the properties of the function $1/(1-x)$ has led, in the past two centuries, to much new mathematics. At the most elementary level, it was noticed long ago that the function $1/(1-x)$ is (roughly) the Laplace transform of the function e^x, and the theory of Laplace transforms, in its heyday, was concerned with drafting a correspondence table between properties of functions and properties of their Laplace transforms. In the development of functional analysis in the forties and fifties, properties of semigroups of linear operators on Banach spaces were thrown back onto properties of their resolvents; the functional equation satisfied by the resolvent of an operator was clearly perceived at that time as an analog—though still a mysterious one—of the Cauchy functional equation satisfied by the exponential function.

A far more ancient, though, it must be admitted, still little known instance of this analogy is the Newton expansion of the calculus of divided difference (see, e.g., [10]). Although Taylor's formula can be formally seen as a special case of a Newton expansion (obtained by taking equal arguments), nonetheless, the algebraic properties of the two expansions were radically different. Suitably reinterpreted, Taylor's formula led in this

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century to the concept of Lie algebra. To the best of our knowledge, no analogous algebraic structure has yet been derived from Newton's expansion, although some recent work of Lascoux and Schützenburger on Schubert polynomials may well foreshadow such a development.

With this possible development in the back of our minds, we undertake in this work the development of what may be called the Newtonian analog of the simplest of the enveloping algebras of a Lie algebra, which is the Hopf algebra $K[x]$ of polynomials in one variable. This Hopf algebra is to the exponential function as the Newtonian coalgebra we study in this work is to the function $1/(1-x)$.

More specifically, following the suggestion of G.-C. Rota [16, p. 753; 7, p. 119], we define a coalgebra structure on the underlying vector space of the algebra $P = K[x]$ by setting

$$\Delta p = \frac{p \otimes 1 - 1 \otimes p}{x \otimes 1 - 1 \otimes x}$$

for every polynomial p in P. The resulting coalgebra, which we call the *Newtonian coalgebra*, is not a bialgebra [17, Chap. III]. Instead, the coproduct Δ satisfies the identity

$$\Delta(pq) = (p \otimes 1)\,\Delta q + \Delta p(1 \otimes q),$$

which is reminiscent of a derivation. To the best of our knowledge, this property was first observed by Joni and Rota [7]. Stretching the meaning of our terms (but only a little), we could say that a Newtonian coalgebra is a coalgebra defined on the underlying vector space of an algebra, where the coproduct is an algebra derivation instead of an algebra endomorphism (as is the case for a bialgebra). (Our definition of a coalgebra differs from that of the standard references [11, 17] in that we do not require a coalgebra to have a counit.) We have not investigated the possibility of extending this definition to the case of several variables (which might give a Newtonian analog of the enveloping algebra of a Lie algebra), but we should like to propose this possibility as an intriguing program.

The bulk of this work is a detailed study of the Newtonian coalgebra. Our main objective is to give a presentation (one that, we would like to believe, meets contemporary standards of rigor) of the algebraic underpinning of the calculus of divided differences. The proofs of Newton's expansion, of the Lagrange interpolation formula, and of their variants and generalizations given below shed light on some classical interpolation problems, at least insofar as the required algebra goes. Our systematic use of the dual algebra brings out the hidden reasons for these interpolation formulas.

In the umbral calculus of Roman and Rota [15], an important role is played by the algebra of shift-invariant operators on the Hopf algebra $K[x]$ (where the comultiplication is defined by making the generator x primitive). It can be shown that the algebra of shift-invariant operators is exactly the algebra of comodule endomorphisms (where we consider $K[x]$ as a comodule over itself) [17, Chap. II]. In parallel with the umbral calculus, we develop the calculus of comodule endomorphisms of P (which we call the algebra of *descending operators*; see Section 6.3), and we show that this algebra plays an analogous role in the Newtonian coalgebra.

Much as the umbral calculus is closely related to properties of polynomial sequences of binomial type, that is, sequences of polynomials $\{p_n\}$ satisfying the identities

$$p_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}p_k(x)\,p_{n-k}(y),$$

we find that the calculus that is associated with the Newtonian coalgebra is related to properties of *Newtonian sequences*, that is, sequences of polynomials $\{p_n\}$ satisfying the identities

$$p_n(x)-p_n(y)=(x-y)\sum_{k=0}^{n-1}p_k(x)\,p_{n-k-1}(y)$$

(see Sect. 5.3). Remarkably, all the main results of the umbral calculus have analogs in the context of Newtonian coalgebras, for example the transfer formula (see Sect. 7.3 and [15, p. 132]). Regrettably, however, few such sequences of polynomials have been studied in the literature, and we have not made a thorough search for those sequences that might appear worth developing from the point of view of the theory of special functions. We hope, however, that the present work will encourage the recognition of such special sequences of polynomials in probabilistic and combinatorial contexts similar to those where sequences of polynomials of binomial type have proved useful.

1.2. *Organization of the Paper*

The paper is organized as follows:

In Section 2, we define the coalgebra structure on P, show that its iterates coincide with the classical divided differences, and derive various technical properties that will be needed later in the paper.

In Section 3, we begin the study of the dual algebra P^*. We define the *Evaluation Algebra* $\mathscr{E}$ to be the subalgebra of P^* generated by the functionals that evaluate a polynomial at a fixed point. We compute the structure of $\mathscr{E}$ as an algebra, and we show that $\mathscr{E}$ consists of all linear combinations of the functionals that evaluate a derivative of a polynomial at a point. We also show that P^* is isomorphic to the algebra of formal power series with zero constant term, and that it is the completion of $\mathscr{E}$ with respect to the topology in which a sequence of functionals approaches zero if its elements are products of an increasing number of factors.

In Section 4, we define a *polynomial sequence* to be a sequence $\{p_n\}$ of polynomials such that p_n is of degree n. We then define a filtration of P^* dual to the filtration of P defined by degree, and define a *functional sequence* to be a sequence of functionals $\{f_1, f_2, f_3, ...\}$ such that f_n is of filtration n, but not of filtration $n+1$. We show that every functional sequence is the dual pseudobasis of a unique polynomial sequence, and we obtain some applications to problems of polynomial interpolation.

In Section 5, we study the group of continuous algebra automorphisms of P^* and the group of coalgebra automorphisms of P, and we show that these groups are anti-isomorphic. We show that every continuous algebra automorphism of P^* is determined by the choice of a *basic functional*, which is an element f of P^* such that $f(1) \neq 0$. (Basic functionals are exactly those elements of P^* whose powers form a functional sequence, and thus a pseudobasis of P^*.) We show that every coalgebra automorphism of P is determined by the choice of a *Newtonian sequence*, which is a polynomial sequence $\{p_n\}$ such that $p_n(x) - p_n(y) = (x-y) \sum_{i=0}^{n-1} p_i(x) p_{n-1-i}(y)$ for $n > 0$. We thus obtain a correspondence between basic functionals and Newtonian sequences.

In Section 6, we study the algebra of *descending operators*. These are linear operators on P defined as the adjoints of certain multiplication operators on P^*, and they will be used later in the paper to construct various polynomial sequences of interest. As an application, we obtain the Lagrange interpolation formula. Among the descending operators are the *basic operators*, which are the adjoints to the operators on P^* defined by multiplication by a basic functional. For each Newtonian sequence $\{p_n\}$, there is a unique basic operator f^* such that $f^*(p_n) = p_{n-1}$. We also show that the algebra of descending operators is exactly the algebra of comodule maps $P \to P$ (where we consider P as a comodule over itself).

In Section 7, we show that Newtonian sequences $\{p_n\}$ are classified by their constant terms, which can be arbitrary except that we must have $p_0 \neq 0$. We characterize Newtonian sequences by their generating functions, and we obtain several constructions of the Newtonian sequence associated with a basic functional. As an application, we obtain an easy proof of the Lagrange inversion formula, and we give a number of examples of Newtonian sequences.

In Section 8, we define a *Scheffer set* to be a polynomial sequence $\{s_n\}$ such that there is a basic operator f^* for which $f^*(s_n) = s_{n-1}$. We show that this is equivalent to there being a Newtonian sequence $\{p_n\}$

such that $s_n(x)-s_n(y)=(x-y)\sum_{i=0}^{n-1} s_i(x)\, p_{n-1-i}(y)$ for $n>0$, and we characterize Sheffer sets by their generating functions. We show that for a basic operator f^* there is one Sheffer set for each invertible descending operator, and that a Sheffer set for f^* is entirely determined by its sequence of constant terms (which can be arbitrary as long as $s_0 \neq 0$).

2. The Newtonian Coalgebra P

2.1. *Definitions and Basic Properties*

In this section, we define the comultiplication of the Newtonian coalgebra P, and show that our comultiplication is a coassociative, cocommutative derivation, for which there is no counit. (Our definition of a coalgebra differs from that of the standard references [11, 17] in that their definition of a coalgebra requires a counit.)

Let K be a field of characteristic zero, and let $P=K[x]$ be the polynomial algebra in one variable over K. There is an isomorphism of algebras between $P\otimes P$ and the polynomial algebra over K in two variables, under which $x\otimes 1$ and $1\otimes x$ map to the two variables. Using this fact, it is easy to see that for each polynomial p there is a unique element q of $P\otimes P$ such that $(x\otimes 1-1\otimes x)q=p\otimes 1-1\otimes p$. We denote this element q by $(p\otimes 1-1\otimes p)/(x\otimes 1-1\otimes x)$.

Definition. Let $\Delta\colon P\to P\otimes P$ be defined by

$$\Delta(p)=\frac{p\otimes 1-1\otimes p}{x\otimes 1-1\otimes x}.$$

Examples. $\Delta(1)=(1\otimes 1-1\otimes 1)/(x\otimes 1-1\otimes x)=0$. *If* $n>0$, *then*

$$\begin{aligned}\Delta(x^n)&=\frac{x^n\otimes 1-1\otimes x^n}{x\otimes 1-1\otimes x}\\&=x^{n-1}\otimes 1+x^{n-2}\otimes x+x^{n-3}\otimes x^2+\cdots+1\otimes x^{n-1}\\&=\sum_{i=0}^{n-1} x^i\otimes x^{n-1-i}.\end{aligned}$$

Proposition 2.1. *The map Δ is linear.*

Proof. Let $\Delta(p)=s$ and let $\Delta(q)=t$, i.e., $(x\otimes 1-1\otimes x)s=p\otimes 1-1\otimes p$ and $(x\otimes 1-1\otimes x)t=q\otimes 1-1\otimes q$; then $(x\otimes 1-1\otimes x)(s+t)=p\otimes 1+q\otimes 1-1\otimes p-1\otimes q=(p+q)\otimes 1+1\otimes(p+q)$. Thus, $\Delta(p+q)=s+t=\Delta p+\Delta q$. If $\lambda\in K$, then $(x\otimes 1-1\otimes x)\lambda s=\lambda(p\otimes 1-1\otimes p)=\lambda p\otimes 1-1\otimes\lambda p$, and so $\Delta(\lambda p)=\lambda(\Delta p)$.

PROPOSITION 2.2. *The comultiplication Δ is coassociative, i.e.,* $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$.

Proof. A straightforward computation shows that

$$(\Delta \otimes 1)\,\Delta(x^n) = \sum_{\substack{i+j+k=n-2 \\ i \geqslant 0,\, j \geqslant 0,\, k \geqslant 0}} x^i \otimes x^j \otimes x^k = (1 \otimes \Delta)\,\Delta(x^n).$$

Since $\{x^n \mid n \geqslant 0\}$ is a basis of P, the result follows.

PROPOSITION 2.3. *The comultiplication Δ is cocommutative, i.e., if $T\colon P \otimes P \to P \otimes P$ is defined by $T(a \otimes b) = b \otimes a$, then $T\Delta = \Delta$.*

Proof. If $n \geqslant 0$, then $T\Delta(x^n) = \Delta(x^n)$, and so $T\Delta$ and Δ agree on a basis of P.

Although P with the comultiplication Δ is a cocommutative coalgebra, we will now show that it is not the coalgebra underlying a Hopf algebra. We will show that there is no counit for Δ, i.e., there is no linear map $\eta\colon P \to K$ such that $(\eta \otimes 1)\Delta = (1 \otimes \eta)\Delta = 1$. We will also show that Δ is not a homomorphism of algebras, but rather that it is a *derivation* in the sense that $\Delta(pq) = (p \otimes 1)\,\Delta q + \Delta p(1 \otimes q)$.

PROPOSITION 2.4. *The comultiplication Δ has no counit.*

Proof. If $\eta\colon P \to K$ is a linear map, then $(\eta \otimes 1)\,\Delta(1) = (\eta \otimes 1)(0 \otimes 0) = 0$, and so $(\eta \otimes 1)\,\Delta(1) \neq 1$. Thus, η is not a counit for Δ.

PROPOSITION 2.5. *The comultiplication Δ is a* derivation *in the sense that*

$$\Delta(pq) = (p \otimes 1)\,\Delta q + \Delta p(1 \otimes q).$$

Proof.

$$\begin{aligned}
\Delta(pq) &= \frac{pq \otimes 1 - 1 \otimes pq}{x \otimes 1 - 1 \otimes x} \\
&= \frac{(p \otimes 1)(q \otimes 1) - (1 \otimes p)(1 \otimes q)}{x \otimes 1 - 1 \otimes x} \\
&= \frac{(p \otimes 1)(q \otimes 1) - (p \otimes 1)(1 \otimes q) + (p \otimes 1)(1 \otimes q) - (1 \otimes p)(1 \otimes q)}{x \otimes 1 - 1 \otimes x} \\
&= (p \otimes 1)\frac{q \otimes 1 - 1 \otimes q}{x \otimes 1 - 1 \otimes x} + \frac{p \otimes 1 - 1 \otimes p}{x \otimes 1 - 1 \otimes x}(1 \otimes q) \\
&= (p \otimes 1)\,\Delta q + \Delta p(1 \otimes q).
\end{aligned}$$

Since Δ is cocommutative, we can also write $\Delta(pq) = (1 \otimes p)\,\Delta q + \Delta p(q \otimes 1)$.

2.2. *Higher Divided Differences*

Just as an algebra multiplication can be composed to yield a multiplication of several factors, a coalgebra comultiplication can be composed to yield a comultiplication to several factors. In this section, we define the higher divided differences to be iterates of Δ, derive some technical properties, and then show that the iterates of Δ coincide with the classical higher divided differences.

DEFINITIONS. We inductively define linear maps $\Delta^i : P \to \bigotimes_{i+1} P$ for $i \geqslant 0$ by letting $\Delta^0 = 1$ (i.e., Δ^0 is the identity map), $\Delta^1 = \Delta$, $\Delta^2 = (\Delta \otimes 1)\Delta$, $\Delta^3 = (\Delta \otimes 1 \otimes 1)(\Delta \otimes 1)\Delta$, and if $n > 3$, $\Delta^n = (\Delta \otimes \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{n-1})\,\Delta^{n-1}$.

Proposition 2.2 implies that we can also write $\Delta^2 = (1 \otimes \Delta)\Delta$, $\Delta^3 = (1 \otimes \Delta \otimes 1)(\Delta \otimes 1)\Delta = \text{etc.}$

We will show at the end of this section that the higher powers of Δ correspond to the classical higher divided differences.

DEFINITION. We define the total degree of $p_1 \otimes p_2 \otimes \cdots \otimes p_n$ to be $\sum_{i=1}^{n}$ (degree of p_i).

PROPOSITION 2.6. *The kernel of Δ consists of the polynomials of degree zero.*

Proof. If $p = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then $\Delta p = a_n$ (terms of total degree $n-1$) $+ a_{n-1}$ (terms of total degree $n-2$) $+ \cdots + a_1$ (terms of total degree 0). Thus, if $a_i \neq 0$ for some $i > 0$, we would have $\Delta p \neq 0$.

PROPOSITION 2.7. *If p is of degree k and $n > k$, then $\Delta^n p = 0$.*

Proof. Since Δ lowers total degree by 1, $\Delta^k p$ is of total degree 0, and so $\Delta^{k+1} p = 0$.

PROPOSITION 2.8. *Let $x_1, x_2, \ldots, x_k$ be an arbitrary set of elements of K. If $n \leqslant k$, then*

$$\begin{aligned}
&\Delta^n((x-x_1)(x-x_2)\cdots(x-x_k)) \\
&\quad = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant k} (x-x_1)\cdots(x-x_{i_1-1}) \\
&\qquad \otimes (x-x_{i_1+1})\cdots(x-x_{i_2-1}) \otimes (x-x_{i_2+1})\cdots(x-x_k),
\end{aligned}$$

where the empty product is defined to equal 1.

A less formal statement is that each term in the sum is obtained by choosing n factors from the set $\{(x-x_1), (x-x_2), ..., (x-x_k)\}$ and then replacing each chosen factor by a tensor product symbol (where we insert a factor of 1 whenever a pair of adjacent factors or one of the end factors is deleted).

Proof. We will prove this proposition by induction on n. To prove the case $n=1$, we will do an induction on k. If $n=1$ and $k=1$, we have $\Delta(x-x_1)=\Delta x-\Delta x_1=1\otimes 1-0\otimes 0=1\otimes 1$. We now assume that the proposition is true for the integer k, i.e.,

$$\begin{aligned}\Delta((x-x_1)(x-x_2)\cdots(x-x_k))&\\ =\sum_{1\leqslant i_1\leqslant k}&(x-x_1)(x-x_2)\cdots(x-x_{i_1-1})\\ &\otimes(x-x_{i_1+1})(x-x_{i_1+2})\cdots(x-x_k).\end{aligned}$$

Using the fact that Δ is a derivation, we have

$$\begin{aligned}&\Delta((x-x_1)(x-x_2)\cdots(x-x_{k+1}))\\ &\quad=\Delta((x-x_1)(x-x_2)\cdots(x-x_k))(x-x_{k+1})\\ &\quad=((x-x_1)(x-x_2)\cdots(x-x_k)\otimes 1)(1\otimes 1)\\ &\qquad+\left(\sum_{1\leqslant i_1\leqslant k}(x-x_1)\cdots(x-x_{i_1-1})\otimes(x-x_{i_1+1})\cdots(x-x_k)\right)\\ &\qquad\times(1\otimes(x-x_{k+1}))\\ &\quad=(x-x_1)(x-x_2)\cdots(x-x_k)\otimes 1\\ &\qquad+\sum_{1\leqslant i_1\leqslant k}(x-x_1)\cdots(x-x_{i_1-1})\otimes(x-x_{i_1+1})\cdots(x-x_{k+1})\\ &\quad=\sum_{1\leqslant i_1\leqslant k+1}(x-x_1)(x-x_2)\cdots(x-x_{i_1-1})\\ &\qquad\otimes(x-x_{i_1+1})(x-x_{i_1+2})\cdots(x-x_{k+1}).\end{aligned}$$

Thus, the proposition is true if $n=1$.

We now assume that the proposition is true for the integer n, and we will prove it for the integer $n+1$.

$$\Delta^{n+1}((x-x_1)(x-x_2)\cdots(x-x_k))$$
$$=(\Delta\otimes\underbrace{1\otimes\cdots\otimes 1}_{n})\,\Delta^n((x-x_1)(x-x_2)\cdots(x-x_k))$$
$$=(\Delta\otimes 1\otimes\cdots\otimes 1)\sum_{1\leqslant i_1<i_2<\cdots<i_n\leqslant k}(x-x_1)(x-x_2)\cdots(x-x_{i_1-1})$$
$$\otimes(x-x_{i_1+1})(x_{i_1+2})\cdots(x-x_k).$$

The terms with $i_1=1$ will go to zero, since $\Delta(1)=0\otimes 0$. Each term with $i_1>1$ will yield the sum

$$\sum_{j=1}^{i_1-1}(x-x_1)(x-x_2)\cdots(x-x_{j-1})\otimes(x-x_{j+1})\cdots(x-x_{i_1-1})$$
$$\otimes(x-x_{i_1+1})\cdots(x-x_k).$$

Since each subset of size $n+1$ of the set $\{1, 2, ..., k\}$ can be constructed in a unique way by first choosing n elements of the set $\{2, 3, ..., k\}$ and then choosing an element of the set $\{1, 2, ..., k\}$ smaller than the elements already chosen, the proof is complete.

For use later in this paper, we record the following fact (whose proof is immediate).

PROPOSITION 2.9. *The number of terms in the sum of the preceding proposition is* $\binom{k}{n}$.

Classically, the divided difference of the function f at the points x_0, x_1 is defined as $[f\colon x_0, x_1]=(f(x_1)-f(x_0))/(x_1-x_0)$, and for $n\geqslant 1$, the $n+1$st divided difference at the points $x_0, x_1, ..., x_{n+1}$ is defined as

$$[f\colon x_0, x_1, ..., x_{n+1}]=\frac{[f\colon x_0, x_1, ..., x_n]-[f\colon x_1, x_2, ..., x_{n+1}]}{x_0-x_{n+1}}$$

(see, e.g., [10]). We will now show that the higher powers of Δ coincide with the classical higher divided differences.

THEOREM 2.10. *If* $n\geqslant 0$, *then*

$$\Delta^{n+1}(p)=\frac{\Delta^n(p)\otimes 1-1\otimes\Delta^n(p)}{x\otimes\underbrace{1\otimes 1\otimes\cdots\otimes 1}_{n+1}-\underbrace{1\otimes 1\otimes\cdots\otimes 1}_{n+1}\otimes x}.$$

Proof. We will prove this theorem by induction on n. The induction is begun by noting that when $n=0$, the theorem is just a restatement of the definition of Δ.

We now assume that the theorem is true for the integer $n-1$; we will show that it is true for the integer n. The induction hypothesis tells us that

$$\Delta^n(p)=\frac{\Delta^{n-1}(p)\otimes 1-1\otimes\Delta^{n-1}(p)}{x\otimes\underbrace{1\otimes 1\otimes\cdots\otimes 1}_{n}-\underbrace{1\otimes 1\otimes\cdots\otimes 1}_{n}\otimes x},$$

i.e., $(x\otimes 1\otimes 1\cdots\otimes 1-1\otimes 1\otimes\cdots\otimes 1\otimes x)\,\Delta^p(p)=\Delta^{n-1}(p)\otimes 1-1\otimes\Delta^{n-1}(p)$. If $\Delta^n(p)=\sum_{(p)} p_1\otimes p_2\otimes\cdots\otimes p_{n+1}$, then this implies that

$$\begin{aligned}\sum_{(p)}&(x\otimes 1\otimes\cdots\otimes 1-1\otimes\cdots\otimes 1\otimes x)(p_1\otimes\cdots\otimes p_{n+1})\\&=\Delta^{n-1}(p)\otimes 1-1\otimes\Delta^{n-1}(p),\end{aligned}$$

i.e.,

$$\begin{aligned}\sum_{(p)}&xp_1\otimes p_2\otimes\cdots\otimes p_{n+1}-\sum_{(p)}p_1\otimes\cdots\otimes p_n\otimes xp_{n+1}\\&=\Delta^{n-1}(p)\otimes 1-1\otimes\Delta^{n-1}(p).\end{aligned}$$

We can now apply $\Delta\otimes\underbrace{1\otimes\cdots\otimes 1}_{n}$ to both sides of this equation, and obtain

$$\begin{aligned}\sum_{(p)}&\Delta(xp_1)\otimes p_2\otimes\cdots\otimes p_{n+1}-\sum_{(p)}\Delta(p_1)\otimes\cdots\otimes p_n\otimes xp_{n+1}\\&=\Delta^n(p)\otimes 1-\Delta(1)\otimes\Delta^{n-1}(p).\end{aligned}$$

Using the facts that Δ is a derivation and $\Delta(1)=0$, we obtain

$$\begin{aligned}\sum_{(p)}&[(x\otimes 1)\,\Delta(p_1)+\Delta(x)(1\otimes p_1)]\otimes p_2\otimes\cdots\otimes p_{n+1}\\&-\sum_{(p)}\Delta(p_1)\otimes\cdots\otimes p_n\otimes xp_{n+1}=\Delta^n(p)\otimes 1.\end{aligned}$$

Since $\Delta(x)=1\otimes 1$, this implies that

$$\begin{aligned}\sum_{(p)}&[(x\otimes 1)\,\Delta(p_1)+(1\otimes p_1)]\otimes p_2\otimes\cdots\otimes p_{n+1}\\&-\sum_{(p)}\Delta(p_1)\otimes p_2\otimes\cdots\otimes p_n\otimes xp_{n+1}=\Delta^n(p)\otimes 1\end{aligned}$$

$$\sum_{(p)} [(x\otimes 1)\Delta(p_1)]\otimes p_2\otimes \cdots \otimes p_{n+1} + \sum_{(p)} 1\otimes p_1\otimes p_2\otimes \cdots \otimes p_{n+1}$$
$$-\sum_{(p)} \Delta(p_1)\otimes p_2\otimes \cdots \otimes p_n\otimes xp_{n+1} = \Delta^n(p)\otimes 1$$

$$\sum_{(p)} [(x\otimes 1)\Delta(p_1)]\otimes p_2\otimes \cdots \otimes p_{n+1}$$
$$-\sum_{(p)} \Delta(p_1)\otimes p_2\otimes \cdots \otimes p_n\otimes xp_{n+1} = \Delta^n(p)\otimes 1 - 1\otimes \Delta^n(p).$$

Writing $\Delta(p_1)=\sum_{(p_1)} p_{11}\otimes p_{12}$, we obtain

$$\sum_{(p)}\sum_{(p_1)} xp_{11}\otimes p_{12}\otimes p_2\otimes \cdots \otimes p_{n+1}$$
$$-\sum_{(p)}\sum_{(p_1)} p_{11}\otimes p_{12}\otimes p_2\otimes \cdots \otimes xp_{n+1} = \Delta^n(p)\otimes 1 - 1\otimes \Delta^n(p)$$

$$(x\otimes \underbrace{1\otimes \cdots \otimes 1}_{n+1} - \underbrace{1\otimes \cdots \otimes 1}_{n+1}\otimes x)$$
$$\times \sum_{(p)}\sum_{(p_1)} p_{11}\otimes p_{12}\otimes p_2\otimes \cdots \otimes p_{n+1} = \Delta^n(p)\otimes 1 - 1\otimes \Delta^n(p)$$

$$(x\otimes \underbrace{1\otimes \cdots \otimes 1}_{n+1} - \underbrace{1\otimes \cdots \otimes 1}_{n+1}\otimes x)(\Delta\otimes \underbrace{1\otimes \cdots \otimes 1}_{n})$$
$$\times \sum_{(p)} p_1\otimes \cdots \otimes p_{n+1} = \Delta^n(p)\otimes 1 - 1\otimes \Delta^n(p)$$

$$(x\otimes 1\otimes \cdots \otimes 1 - 1\otimes \cdots \otimes 1\otimes x)[(\Delta\otimes 1\otimes \cdots \otimes 1)\Delta^n(p)]$$
$$= \Delta^n(p)\otimes 1 - 1\otimes \Delta^n(p)$$

$$(\Delta\otimes 1\otimes \cdots \otimes 1)\Delta^n(p) = \frac{\Delta^n(p)\otimes 1 - 1\otimes \Delta^n(p)}{x\otimes 1\otimes \cdots \otimes 1 - 1\otimes \cdots \otimes 1\otimes x}$$

$$\Delta^{n+1}(p) = \frac{\Delta^n(p)\otimes 1 - 1\otimes \Delta^n(p)}{x\otimes 1\otimes \cdots \otimes 1 - 1\otimes \cdots \otimes 1\otimes x}.$$

This completes the induction step, and so the proof is complete.

3. The Dual Algebra

3.1. *The Evaluation Algebra*

The comultiplication Δ on P gives the dual space P^* the structure of a (non-unitary) commutative algebra. In this section, we show that the subalgebra $\mathscr{E}$ of P^* generated by the evaluation functionals consists of all linear combinations of derivative functionals, and we compute the structure of $\mathscr{E}$. In the next section, we will show that P^* is the completion of $\mathscr{E}$ with respect to the topology in which the sequence $(f_1, f_2, f_3, \ldots)$ approaches zero if the f_i are products of an increasing number of factors.

The dual of the comultiplication Δ is a linear map $\Delta^*\colon (P \otimes P)^* \to P^*$. If we compose this with the natural inclusion $P^* \otimes P^* \subset (P \otimes P)^*$, we obtain a linear map (which, by an abuse of notation, we will also call Δ^*) $\Delta^*\colon P^* \otimes P^* \to P^*$.

THEOREM 3.1. *The map Δ^* gives P^* the structure of a (non-unitary) commutative algebra.*

Proof. The only things requiring proof are that the multiplication is associative and commutative. To show that the multiplication is associative, let $f, g, h \in P^*$ and let $p \in P$; then

$$\begin{aligned}
((fg)h)\, p &= [\Delta^*((\Delta^*(f \otimes g)) \otimes h)]\, p \\
&= ((\Delta^*(f \otimes g)) \otimes h)(\Delta p) \\
&= (f \otimes g \otimes h)((\Delta \otimes 1)(\Delta p)). \\
&= (f \otimes g \otimes h)((1 \otimes \Delta)(\Delta p)) \qquad \text{(using Proposition 2.2)} \\
&= (f \otimes (\Delta^*(g \otimes h)))(\Delta p) \\
&= [\Delta^*(f \otimes (\Delta^*(g \otimes h)))]\, p \\
&= (f(gh))\, p.
\end{aligned}$$

Thus, the multiplication is associative.

To show that the multiplication is commutative, let $f, g \in P^*$ and let $p \in P$; then

$$\begin{aligned}
(fg)\, p &= (f \otimes g)\, \Delta p \\
&= (f \otimes g)\, T \Delta p \qquad \text{(using Proposition 2.3)} \\
&= (g \otimes f)\, \Delta p \\
&= (gf)\, p,
\end{aligned}$$

and the proof is complete.

DEFINITIONS. If $a \in K$, we let ε_a denote the element of P^* that evaluates the polynomial p on a, i.e., $\varepsilon_a(p) = p(a)$. Since ε_0 is just the augmentation of the algebra P, we will use ε to denote ε_0. We will call $\{\varepsilon_a \mid a \in K\}$ the *evaluation functionals*, and the subalgebra of P^* generated by the evaluation functionals the *evaluation algebra* $\mathscr{E}$.

Notation. If $x_1, x_2, x_3, \ldots$ are elements of K, we will often let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ denote $\varepsilon_{x_1}, \varepsilon_{x_2}, \varepsilon_{x_3}, \ldots$.

DEFINITION. We will call a functional of the form $\varepsilon_a D^k$ for $a \in K$ and $k \geqslant 0$ a *derivative functional.*

We will now show that the evaluation algebra contains all derivative functionals.

PROPOSITION 3.2. *If* $a \in K$ *and* $k > 0$, *then*

$$\varepsilon_a^k = \frac{1}{(k-1)!} \varepsilon_a D^{k-1}.$$

Proof. We will show that the two functionals of the proposition agree on the basis $\{x^n\}$ of P. If $n > k-1$, then Proposition 2.8 implies that

$$\begin{aligned} \varepsilon_a^k(x^n) &= \underbrace{(\varepsilon_a \otimes \cdots \otimes \varepsilon_a)}_{k}(\Delta^{k-1} x^n) \\ &= (\varepsilon_a \otimes \cdots \otimes \varepsilon_a) \sum x^{i_1} \otimes \cdots \otimes x^{i_k} \\ &= \sum a^{i_1} a^{i_2} \cdots a^{i_k} \end{aligned}$$

(where the sum is over all sequences of non-negative integers $(i_1, i_2, \ldots, i_k)$ such that $i_1 + i_2 + \cdots + i_k = n-(k-1)$). Proposition 2.9 implies that this equals

$$\binom{n}{k-1} a^{n-(k-1)} = \frac{1}{(k-1)!} \varepsilon_a D^{k-1}(x^n).$$

If $n = k-1$, then Proposition 2.8 implies that

$$\begin{aligned} \varepsilon_a^k(x^n) &= \underbrace{(\varepsilon_a \otimes \cdots \otimes \varepsilon_a)}_{k} \underbrace{(1 \otimes \cdots \otimes 1)}_{k} \\ &= 1 \\ &= \frac{1}{(k-1)!} \varepsilon_a D^{k-1}(x^n). \end{aligned}$$

Finally, if $n < k-1$, then Proposition 2.7 implies that $\varepsilon_a^k(x^n) = (\varepsilon_a \otimes \cdots \otimes \varepsilon_a)\, \Delta^{k-1}(x^n) = 0$, and so the proof is complete.

If $a \neq b$, then $\varepsilon_a \varepsilon_b(x^n) = \sum_{i=0}^{n-1} a^i b^{n-1-i} = (a^n - b^n)/(a-b) = (\varepsilon_a/(a-b) + \varepsilon_b/(b-a))(x^n)$. Thus, if $a \neq b$, $\varepsilon_a \varepsilon_b = \varepsilon_a/(a-b) + \varepsilon_b/(b-a)$. Using this relation and a straightforward induction argument, one can show that the product $\varepsilon_1^{n_1} \varepsilon_2^{n_2} \cdots \varepsilon_k^{n_k}$ (where the x_i are distinct elements of K) can be written as a linear combination of the functionals ε_i^j for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n_i$. Thus, we have proved the following.

Proposition 3.3. *The evaluation algebra $\mathscr{E}$ consists of all linear combinations of derivative functionals.*

We will now show that $\varepsilon_a \varepsilon_b = \varepsilon_a/(a-b) + \varepsilon_b/(b-a)$ for $a \neq b$ is the only relation that holds in $\mathscr{E}$, i.e., that $\mathscr{E}$ is the free non-unitary algebra in the $\{\varepsilon_a\}$ divided by the ideal generated by $\{\varepsilon_a \varepsilon_b - \varepsilon_a/(a-b) - \varepsilon_b/(b-a) \mid a \neq b\}$. (The free non-unitary algebra in the $\{\varepsilon_a\}$ is naturally isomorphic to the augmentation ideal of the polynomial algebra in the $\{\varepsilon_a\}$. This is just the algebra of polynomials in the $\{\varepsilon_a\}$ with constant term equal to zero.) We will show in the next section that the algebra P^* is the completion of ε with respect to the topology in which the sequence $(f_1, f_2, f_3, \ldots)$ approaches zero if the f_i are products of an increasing number of factors.

Theorem 3.4. *The evaluation algebra $\mathscr{E}$ is isomorphic to the free non-unitary algebra in the ε_a (i.e., the algebra of polynomials in the ε_a with zero constant term) divided by the ideal generated by $\{\varepsilon_a \varepsilon_b - \varepsilon_a/(a-b) - \varepsilon_b/(b-a) \mid a \neq b\}$.*

The proof of Theorem 3.4 will use the following lemma, which we will prove in Section 4.2.

Lemma 3.5. *The $\{\varepsilon_a^j\}$ are a linearly independent subset of P^*.*

Assuming this lemma, we are now able to prove Theorem 3.4.

Proof of Theorem 3.4. Let A be the free non-unitary algebra over K generated by the symbols $\{e_a \mid a \in K\}$ and let I be the ideal of A generated by $\{e_b e_b - e_a/(a-b) - e_b/(b-a) \mid a \neq b\}$. Define a homomorphism $f\colon A \to P^*$ by letting $f(e_a) = \varepsilon_a$. The image of f is $\mathscr{E}$. The ideal I is clearly contained in the kernel of f, and so it remains only to show that the kernel is contained in I.

Let $v \in \text{kernel}(f)$. If v has any terms that involve a product $e_a e_b$ with $a \neq b$, then, by adding an element of I to v, we can replace the $e_a e_b$ by $e_a/(a-b) + e_b/(b-a)$. After a finite number of these operations, we will have replaced v with another element of kernel(f) which is a linear combination of terms of the form e_a^j. Since the $\{\varepsilon_a^j\}$ are a linearly independent set in P^*, this implies that by adding an element of I to v, we have reduced v to zero. Thus, $v \in I$, and the proof is complete.

3.2. *Formal Power Series*

In this section, we show that P^* is isomorphic to the algebra of formal power series with zero constant term. This will imply that P^* is the completion of $\mathscr{E}$ with respect to the topology in which the sequence $(f_1, f_2, f_3, ...)$ approaches zero if the f_i are products of an increasing number of factors.

If $f \in P^*$ and n is a positive integer, then Proposition 2.7 implies that $f^{n+1} = (f \otimes \cdots \otimes f)\,\Delta^n$ vanishes on polynomials of degree less than n. Thus, if $\{a_0, a_1, a_2, ...\}$ is a sequence in K, the formal infinite sum $a_0 f + a_1 f^2 + a_2 f^3 + \cdots = \sum_{i=1}^{\infty} a_{i-1} f^i$, when applied to a polynomial p, gives a finite sum. Thus, $\sum_{i=1}^{\infty} a_{i-1} f^i$ is a well-defined element of P^*.

LEMMA 3.6. *If $f \in P^*$, then $f = \sum_{i=1}^{\infty} f(x^{i-1})\,\varepsilon^i$.*

Proof. Since $\varepsilon^{n+1}(x^k) = (1/n!)\,\varepsilon D^n(x^k) = \delta_{n,k}$, the formal power series $f(1)\varepsilon + f(x)\,\varepsilon^2 + f(x^2)\,\varepsilon^3 + \cdots$ agrees with f on each x^n. Since $\{1, x, x^2, ...\}$ is a basis of P, this implies that $f = f(1)\varepsilon + f(x)\,\varepsilon^2 + f(x^2)\,\varepsilon^3 + \cdots$ on all of P.

We will now show that this representation of elements of P^* as formal power series is actually an isomorphism of algebras.

THEOREM 3.7. *If FPS_0 denotes the algebra of formal power series in the variable t with zero constant term (i.e., $FPS_0 = \{a_0 t + a_1 t^2 + a_2 t^3 + \cdots \mid a_i \in K\}$), then $P^* \cong FPS_0$ as algebras.*

Proof. Define $\psi\colon P^* \to \mathrm{FPS}_0$ by $\psi(f) = \sum_{i=1}^{\infty} f(x^{i-1})\,t^i$, i.e., $\psi(f) = f(1)t + f(x)\,t^2 + f(x^2)\,t^3 + \cdots$. The function ψ is a homomorphism because

$$\begin{aligned}\psi(fg) &= \sum_{i=1}^{\infty} ((fg)\,x^{i-1})\,t^i \\ &= \sum_{i=2}^{\infty} ((f \otimes g)\,\Delta x^{i-1})\,t^i \\ &= \sum_{i=2}^{\infty} \left(\sum_{j=0}^{i-2} f(x^j)\,g(x^{i-2-j}) \right) t^i\end{aligned}$$

$$= \left(\sum_{i=1}^{\infty} f(x^{i-1})\, t^i \right) \left(\sum_{i=1}^{\infty} g(x^{i-1})\, t^i \right)$$
$$= \psi(f)\, \psi(g).$$

The homomorphim ψ is injective because if $\psi(f) = 0$, then $f(x^i) = 0$ for all $i \geqslant 0$, and so $f = 0$. Finally, ψ is surjective because if $a_0 t + a_1 t^2 + a_2 t^3 + \cdots$ is an arbitrary element of FPS_0, then we can define $f \in P^*$ by $f(x^n) = a_n$, and we will have $\psi(f) = a_0 t + a_1 t^2 + a_2 t^3 + \cdots$.

COROLLARY 3.8. *If $f \in P^*$, then f vanishes on all polynomials of degree less than $n-1$ if and only if f can be written as a product of n factors.*

Proof. If $f = f_1 f_2 \cdots f_n$, then $f(p) = (f_1 \otimes f_2 \otimes \cdots \otimes f_n)\, \Delta^{n-1} p$. If the degree of p is less than $n-1$, then Proposition 2.7 implies that $\Delta^{n-1} p = 0$.

Conversely, if f vanishes on all polynomials of degree less than $n-1$, then $f(x^k) = 0$ for $0 \leqslant k \leqslant n-2$, and so $f = f(x^{n-1})\, \varepsilon^n + f(x^n)\, \varepsilon^{n+1} + \cdots$. Thus, $f = \varepsilon^{n-1}(f(x^{n-1})\varepsilon + f(x^n)\, \varepsilon^2 + \cdots)$.

We can now show that P^* is the completion of $\mathscr{E}$ with respect to the topology in which the sequence $(f_1, f_2, f_3, \ldots)$ approaches zero if the f_i are the products of an increasing number of factors. We define a filtration $\mathscr{E} = F_1 \supset F_2 \supset F_3 \supset \cdots$ of $\mathscr{E}$ by letting $F_n = \{ f \in \mathscr{E} \mid f$ can be written as a product of n elements of $\mathscr{E}\}$. We topologize $\mathscr{E}$ by taking the F_n as basic neighborhoods of zero (see, e.g., [2, Chap. 10]). If we let I equal the ideal of $\mathscr{E}$ generated by ε, then $I = \mathscr{E}$ and $F_n = I^n$. Theorem 3.7 now implies the following.

THEOREM 3.9. *The algebra P^* is the completion of the evaluation algebra $\mathscr{E}$.*

3.3. *Expanding Products of the ε_a*

In this section, we show how to expand a product of evaluation functionals as a linear combination of derivative functionals.

As we remarked earlier, the relation $\varepsilon_a \varepsilon_b = \varepsilon_a/(a-b) + \varepsilon_b/(b-a)$ for $a \neq b$ implies that the product $\varepsilon_1^{n_1} \varepsilon_2^{n_2} \cdots \varepsilon_k^{n_k}$ (where the x_i are distinct elements of K) can be written as a linear combination of the functionals ε_i^j for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n_i$. Since $\varepsilon_a^k = (\varepsilon_a D^{k-1})/(k-1)!$, it will be important to know exactly how to do this. Since stating the theorem in full generality would involve a confusion of indices, we will state and prove a particular case that makes the general statement clear.

THEOREM 3.10. *If x_1, x_2, and x_3 are distinct elements of K, then*

$$\varepsilon_1^3\varepsilon_2\varepsilon_3^2 = \frac{\begin{vmatrix} \varepsilon_1 & \varepsilon_1^2 & \varepsilon_1^3 & \varepsilon_2 & \varepsilon_3 & \varepsilon_3^2 \\ x_1^4 & 4x_1^3 & 6x_1^2 & x_2^4 & x_3^4 & 4x_3^3 \\ x_1^3 & 3x_1^2 & 3x_1 & x_2^3 & x_3^3 & 3x_3^2 \\ x_1^2 & 2x_1 & 1 & x_2^2 & x_3^2 & 2x_3 \\ x_1 & 1 & 0 & x_2 & x_3 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} x_1^5 & 5x_1^4 & 10x_1^3 & x_2^5 & x_3^5 & 5x_3^4 \\ x_2^4 & 4x_1^3 & 6x_1^2 & x_2^4 & x_3^4 & 4x_3^3 \\ x_1^3 & 3x_1^2 & 3x_1 & x_2^3 & x_3^3 & 3x_3^2 \\ x_1^2 & 2x_1 & 1 & x_2^2 & x_3^2 & 2x_3 \\ x_1 & 1 & 0 & x_2 & x_3 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{vmatrix}}.$$

In general, the expression for $\varepsilon_1^{n_1}\varepsilon_2^{n_2}\cdots\varepsilon_k^{n_k}$, where $n=\sum_{i=1}^k n_i$, is a quotient of two n by n determinants. Each column of the denominator is obtained by evaluating a functional on x^i for $0 \leqslant i \leqslant n-1$. The functionals that are used are $\varepsilon_1, \varepsilon_1^2, \ldots, \varepsilon_1^{n_1}$ followed by $\varepsilon_2, \varepsilon_2^2, \ldots, \varepsilon_2^{n_2}$ and so on up to $\varepsilon_k^{n_k}$. The determinant in the numerator is obtained from the one in the denominator by replacing the top entry in each column by the functional whose values determined that column.

Proof. We would like to find a_{11}, a_{12}, a_{13}, a_2, a_{31}, and a_{32} in P such that

$$\varepsilon_1^3\varepsilon_2\varepsilon_3^2 = a_{11}\varepsilon_1 + a_{12}\varepsilon_1^2 + a_{13}\varepsilon_1^3 + a_2\varepsilon_2 + a_{31}\varepsilon_3 + a_{32}\varepsilon_3^2.$$

Since Proposition 2.7 and Proposition 2.8 imply that

$$(\varepsilon_1^3\varepsilon_2\varepsilon_3^2)(x^n) = \begin{cases} 0, & \text{if } 0 \leqslant n \leqslant 4, \\ 1, & \text{if } n=5, \end{cases}$$

we have the system of equations

$$\begin{pmatrix} x_1^5 & 5x_1^4 & 10x_1^3 & x_2^5 & x_3^5 & 5x_3^4 \\ x_1^4 & 4x_1^3 & 6x_1^2 & x_2^4 & x_3^4 & 4x_3^3 \\ x_1^3 & 3x_1^2 & 3x_1 & x_2^3 & x_3^3 & 3x_3^2 \\ x_1^2 & 2x_1 & 1 & x_2^2 & x_3^2 & 2x_3 \\ x_1 & 1 & 0 & x_2 & x_3 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_2 \\ a_{31} \\ a_{32} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If we now use Cramer's rule to solve for the coefficients, we obtain, e.g.,

$$a_{11}=\frac{\begin{vmatrix} 1 & 5x_1^4 & 10x_1^3 & x_2^5 & x_3^5 & 5x_3^4 \\ 0 & 4x_1^3 & 6x_1^2 & x_2^4 & x_3^4 & 4x_3^3 \\ 0 & 3x_1^2 & 3x_1 & x_2^3 & x_3^3 & 3x_3^2 \\ 0 & 2x_1 & 1 & x_2^2 & x_3^2 & 2x_3 \\ 0 & 1 & 0 & x_2 & x_3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} x_1^5 & 5x_1^4 & 10x_1^3 & x_2^5 & x_3^5 & 5x_3^4 \\ x_1^4 & 4x_1^3 & 6x_1^2 & x_2^4 & x_3^4 & 4x_3^3 \\ x_1^3 & 3x_1^2 & 3x_1 & x_2^3 & x_3^3 & 3x_3^2 \\ x_1^2 & 2x_1 & 1 & x_2^2 & x_3^2 & 2x_3 \\ x_1 & 1 & 0 & x_2 & x_3 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{vmatrix}}.$$

If we expand the upper determinant of

$$\frac{\begin{vmatrix} \varepsilon_1 & \varepsilon_1^2 & \varepsilon_1^3 & \varepsilon_2 & \varepsilon_3 & \varepsilon_3^2 \\ x_1^4 & 4x_1^3 & 6x_1^2 & x_2^4 & x_3^4 & 4x_3^3 \\ x_1^3 & 3x_1^2 & 3x_1 & x_2^3 & x_3^3 & 3x_3^2 \\ x_1^2 & 2x_1 & 1 & x_2^2 & x_3^2 & 2x_3 \\ x_1 & 1 & 0 & x_2 & x_3 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} x_1^5 & 5x_1^4 & 10x_1^3 & x_2^5 & x_3^5 & 5x_3^4 \\ x_1^4 & 4x_1^3 & 6x_1^2 & x_2^4 & x_3^4 & 4x_3^3 \\ x_1^3 & 3x_1^2 & 3x_1 & x_2^3 & x_3^3 & 3x_3^2 \\ x_1^2 & 2x_1 & 1 & x_2^2 & x_3^2 & 2x_3 \\ x_1 & 1 & 0 & x_2 & x_3 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{vmatrix}}$$

along the top row, the result follows.

In the case in which each ε_i is raised to the first power, this theorem takes a particularly simple form.

COROLLARY 3.11. *If $x_1, x_2, ..., x_n$ are distinct elements of K, then*

$$\varepsilon_1\varepsilon_2\cdots\varepsilon_n=\frac{\varepsilon_1}{\prod_{i\neq 1}(x_1-x_i)}+\frac{\varepsilon_2}{\prod_{i\neq 2}(x_2-x_i)}+\cdots+\frac{\varepsilon_n}{\prod_{i\neq n}(x_n-x_i)}.$$

Proof. Theorem 3.10 gives us

$$\varepsilon_1\varepsilon_2\cdots\varepsilon_n=\frac{\begin{vmatrix}\varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n\\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2}\\ x_1^{n-3} & x_2^{n-3} & \cdots & x_n^{n-3}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & 1\end{vmatrix}}{\begin{vmatrix}x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}\\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2}\\ x_1^{n-3} & x_2^{n-3} & \cdots & x_n^{n-3}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & 1\end{vmatrix}}.$$

If we reverse the order of the rows in both the numerator and denominator, the value of the quotient of the determinants will be unchanged. Thus,

$$\varepsilon_1\varepsilon_2\cdots\varepsilon_n=\frac{\begin{vmatrix}1 & 1 & \cdots & 1\\ x_1 & x_2 & \cdots & x_n\\ \vdots & \vdots & \ddots & \vdots\\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2}\\ \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n\end{vmatrix}}{\begin{vmatrix}1 & 1 & \cdots & 1\\ x_1 & x_2 & \cdots & x_n\\ x_1^2 & x_2^2 & \cdots & x_n^2\\ \vdots & \vdots & \ddots & \vdots\\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}\end{vmatrix}}.$$

The determinant in the denominator is now a Vandermonde determinant, whose value is

$$\prod_{i>j}(x_i-x_j).$$

If we expand the numerator along the last row, we get the sum of $(-1)^{k+n}\varepsilon_k$ times a Vandermonde determinant whose value is

$$\prod_{\substack{i>j\\ i,j\neq k}}(x_i-x_j).$$

Thus,

$$\begin{aligned}
\varepsilon_1\varepsilon_2\cdots\varepsilon_n &= \sum_{k=1}^{n} \frac{(-1)^{k+n}(\prod_{i>j,\, i,j\neq k}(x_i-x_j))\,\varepsilon_k}{\prod_{i>j}(x_i-x_j)} \\
&= \sum_{k=1}^{n} \frac{(-1)^{k+n}\varepsilon_k}{(\prod_{i=1}^{k-1}(x_k-x_i))(\prod_{i=k+1}^{n}(x_i-x_k))} \\
&= \sum_{k=1}^{n} \frac{(-1)^{k+n+n-k}\varepsilon_k}{\prod_{i\neq k}(x_k-x_i)} \\
&= \sum_{k=1}^{n} \frac{\varepsilon_k}{\prod_{i\neq k}(x_k-x_i)}.
\end{aligned}$$

Thus,

$$\varepsilon_1\varepsilon_2\cdots\varepsilon_n = \frac{\varepsilon_1}{\prod_{i\neq 1}(x_1-x_i)} + \frac{\varepsilon_2}{\prod_{i\neq 2}(x_2-x_i)} + \cdots + \frac{\varepsilon_n}{\prod_{i\neq n}(x_n-x_i)}.$$

4. Bases and Pseudobases

4.1. *Polynomial Sequences and Functional Sequences*

In this section, we define a filtration of P^* dual to the filtration of P defined by degree. Given a sequence of polynomials with one polynomial of each degree, its "dual basis" is a pseudobasis of P^*, and it consists of a sequence of functionals of increasing filtration. We show here that every such sequence of functionals arises in this way.

Definition. A *polynomial sequence* is a sequence $\{p_n\}$ of elements of P such that p_n is of degree n.

We will make frequent use of the fact that a polynomial sequence forms a basis of P.

We now define a filtration of P^* dual to the filtration of P defined by degree. (This will be the same filtration that we used to define the topology on $\mathscr{E}$.) We define a sequence $P^* = F_1 \supset F_2 \supset F_3 \supset \cdots$ of subalgebras of P^* by letting $F_n = \{f \in P^* \mid f \text{ can be written as a product of } n \text{ elements of } P^*\}$. An element of F_n is said to be of *filtration n*. Corollary 3.8 implies that a functional is of filtration n if and only if it vanishes on all polynomials of degree less than $n-1$.

Definition. A *functional sequence* is a sequence $\{f_1, f_2, f_3, \ldots\}$ of elements of P^* such that f_n is of filtration n, but not of filtration $n+1$.

Every polynomial sequence $\{p_k\}$ has a functional sequence associated with it, in the following way: If $n \geqslant 0$, define f_{n+1} by letting $f_{n+1}(p_k) = \delta_{n,k}$. Since $\{p_0, p_1, ..., p_{n-1}\}$ is a basis for the space of polynomials of degree less than n, f_{n+1} vanishes on that space, and so Corollary 3.8 implies that f_{n+1} is of filtration $n+1$. Since $f_{n+1}(p_n) \neq 0$, f_{n+1} is not of filtration $n+2$, and so the sequence $\{f_1, f_2, f_3, ...\}$ is a functional sequence. We will now show that this in fact defines a one-to-one correspondence between the set of polynomial sequences and the set of functional sequences.

PROPOSITION 4.1. *The above construction defines a one-to-one correspondence between the set of polynomial sequences and the set of functional sequences.*

Proof. Let $\{f_1, f_2, f_3, ...\}$ be a functional sequence; we must show that there is a unique polynomial sequence $\{p_k\}$ such that $f_{n+1}(p_k) = \delta_{n,k}$. Since each f_n is of filtration n but not of filtration $n+1$, we can write $f_n = f_{n,n-1}\varepsilon^n + f_{n,n}\varepsilon^{n+1} + f_{n,n+1}\varepsilon^{n+2} + \cdots$ where each $f_{n,n-1} \neq 0$. In order to find p_k, we can write $p_k = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$ and then solve the system of equations

$$\begin{pmatrix} f_{1,0} & f_{1,1} & f_{1,2} & \cdots & f_{1,k} \\ 0 & f_{2,1} & f_{2,2} & \cdots & f_{2,k} \\ 0 & 0 & f_{3,2} & \cdots & f_{3,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & f_{k+1,k} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The coefficient matrix is an upper triangular matrix with non-zero terms along the diagonal. Thus, the determinant of the matrix is non-zero, and so there is a unique solution to the system. Thus, for each k, there is a unique polynomial p_k of degree k satisfying $f_{n+1}(p_k) = \delta_{n,k}$, and the proof is complete.

DEFINITION. We will call the polynomial sequence constructed in the above proof the polynomial sequence *associated* with the functional sequence $\{f_1, f_2, f_3, ...\}$.

COROLLARY 4.2. *Every functional sequence forms a pseudobasis of P^*.*

Proof. Let $\{f_1, f_2, f_3, ...\}$ be a functional sequence; we must show that every element of P^* has a unique expression in the form $a_1 f_1 + a_2 f_2 + a_3 f_3 + \cdots$. Proposition 4.1 implies that there is a polynomial sequence $\{p_n\}$ such that $f_{n+1}(p_k) = \delta_{n,k}$. If $g \in P^*$, then $\sum_{i=0}^{\infty} g(p_i) f_{i+1}$ agrees with g on the basis $\{p_n\}$ of P, and so $g=\sum_{i=1}^{\infty} g(p_{i-1}) f_i$. If we also have $g=\sum_{i=1}^{\infty} a_i f_i$, then (applying both sides to p_n) we have $g(p_{n-1})=a_i$, and so the proof is complete.

4.2. *The Generalized Newton Interpolation Formula*

In this section, we show that every functional sequence determines an interpolation formula both for functionals and for operators.

THEOREM (Generalized Newton Interpolation Formula) 4.3. *Let $\{p_k\}$ be a polynomial sequence, and let $\{f_k\}$ be its associated functional sequence. If F is either an element of P^* or an operator on P, then*

$$F=\sum_{i=0}^{\infty} F(p_i) f_{i+1}.$$

Proof. It is sufficient to show that the two sides of the equation agree on the basis $\{p_k\}$ of P. Since $f_{i+1}(p_k)=0$ for $i\neq k$, if we apply the right-hand side to the polynomial p_k, the only non-zero term is $F(p_k) f_{k+1}(p_k)=F(p_k)$, and so the proof is complete.

Let $\{x_1, x_2, x_3, ...\}$ be an arbitrary sequence in K. The *Newton polynomials* are the polynomials in the sequence $p_n=\prod_{i=1}^{n} (x-x_i)$. If we use as our polynomial sequence the Newton polynomials, we obtain the classical Newton Interpolation Formula.

LEMMA 4.4. *The functional sequence $\{\varepsilon_1, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_2\varepsilon_3, ...\}$ is the functional sequence associated with the polynomial sequence consisting of the Newton polynomials $\{1, (x-x_1), (x-x_1)(x-x_2), (x-x_1)(x-x_2)(x-x_3), ...\}$.*

Proof. We must show that $\varepsilon_1\varepsilon_2\cdots\varepsilon_{n+1}((x-x_1)(x-x_2)\cdots(x-x_k))=\delta_{n,k}$. Since

$$\begin{aligned}&\varepsilon_1\varepsilon_2\cdots\varepsilon_{n+1}((x-x_1)(x-x_2)\cdots(x-x_k))\\&\quad=(\varepsilon_1\otimes\varepsilon_2\otimes\cdots\otimes\varepsilon_{n+1})\,\Delta^n((x-x_1)(x-x_2)\cdots(x-x_k)),\end{aligned}$$

if $n>k$, then the result follows from Proposition 2.7. If $n=k$, then Proposition 2.8 implies that $\Delta^n((x-x_1)(x-x_2)\cdots(x-x_k))=1\otimes\cdots\otimes 1$, and so $\varepsilon_1\varepsilon_2\cdots\varepsilon_{n+1}((x-x_1)(x-x_2)\cdots(x-x_k))=1$. If $n<k$, then Proposition 2.8 implies that $\Delta^n((x-x_1)(x-x_2)\cdots(x-x_k))$ is a sum of terms, each of which has at least one $(x-x_i)$ in the ith factor. Since $\varepsilon_i(x-x_i)=0$, this completes the proof.

COROLLARY (Newton Interpolation Formula) 4.5. *If $\{x_1, x_2, x_3, ...\}$ is an arbitrary sequence in K, then*

$$\mathrm{Id} = \varepsilon_1 + \sum_{i=1}^{\infty} (x-x_1)(x-x_2)\cdots(x-x_i)\,\varepsilon_1\varepsilon_2\cdots\varepsilon_{i+1}$$

as operators on P.

Proof. This follows from applying Theorem 4.3 to the identity operator, using Lemma 4.4.

The following corollary shows how we can use Theorem 4.3 to find a polynomial fulfilling particular requirements.

COROLLARY 4.6. *If $\{f_1, f_2, f_3, ...\}$ is a functional sequence, n is a positive integer, and $w_1, w_2, w_3, ..., w_n$ are elements of K, then there exists a unique polynomial p of degree less than or equal to $n-1$ for which $f_i(p) = w_i$ for $1 \leqslant i \leqslant n$.*

Proof. Let $\{p_k\}$ be the polynomial sequence associated with the functional sequence $\{f_1, f_2, f_3, ...\}$, and let $p = \sum_{i=1}^{n} w_i p_{i-1}$. Since each p_k is of degree k, p is of degree less than or equal to $n-1$, and since $f_{k+1}(p_i) = \delta_{k,i}$, we have $f_i(p) = w_i$ for $1 \leqslant i \leqslant n$.

Let q be any polynomial of degree less than or equal to $n-1$ for which $f_i(q) = w_i$ for $1 \leqslant i \leqslant n$; we must show that $q = p$. If we apply Theorem 4.3 (using the functional sequence $\{f_1, f_2, f_3, ...\}$ and the polynomial sequence $\{p_k\}$) to the identity operator, we obtain the interpolation formula $\mathrm{Id} = \sum_{i=1}^{\infty} p_{i-1} f_i$. If we apply this to q, then we obtain

$$q = \sum_{i=1}^{\infty} p_{i-1} f_i(q) = \sum_{i=1}^{n} p_{i-1} f_i(q) = \sum_{i=1}^{n} w_i p_{i-1} = p,$$

and the proof is complete.

As an example of the use of Corollary 4.6, suppose that we wish to find a polynomial having predetermined values for some of its derivatives at some finite set of points. (This is a special case of the Birkhoff– Hermite problem; see [9].) To do this, we will apply Corollary 4.6 to the functional sequence $\{\varepsilon_1, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_2\varepsilon_3, ...\}$.

COROLLARY 4.7. *Given a finite set of points $\{a_i\}_{i=1}^{n}$ and, for each a_i, a finite set of values $\{v_{i0}, v_{i1}, v_{i2}, ..., v_{ik_i}\}$, there exists a unique polynomial p of degree less than or equal to $d = [\sum k_i] + n - 1$ such that $D^j p|_{a_i} = v_{ij}$ for $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant k_i$.*

Proof. Let $x_1, x_2, ..., x_{k_1+1} = a_1$, $x_{k_1+2}, ..., x_{k_1+k_2+2} = a_2, ..., x_d = a_n$, and choose x_i arbitrarily for $i > d$. In order to apply Corollary 4.6 to the functional sequence $\{\varepsilon_1, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_2\varepsilon_3, ...\}$, we must show that there is a unique choice of values of $\varepsilon_1\varepsilon_2\cdots\varepsilon_j(p)$ that will produce the desired values of $D^jp|_{a_i}=\varepsilon_{a_i}D^jp$. If we use Theorem 3.10 to write each $\varepsilon_1\varepsilon_2\cdots\varepsilon_j$ as a linear combination of the $\varepsilon_i^k=(\varepsilon_i/(k-1)!)\,D^{k-1}$, we see that for $1\leqslant j\leqslant d$, if $x_j=a_i$, then $\varepsilon_1\varepsilon_2\cdots\varepsilon_j$ is a linear combination of the functionals $\varepsilon_{a_1}, \varepsilon_{a_1}^2, ..., \varepsilon_{a_1}^{k_1+1}$, ε_{a_2}, $\varepsilon_{a_2}^2, ...$, $\varepsilon_{a_2}^{k_2+1}$, ..., $\varepsilon_{a_{i-1}}^{k_{i-1}+1}$, ε_{a_i}, $\varepsilon_{a_i}^2, ...,$ $\varepsilon_{a_i}^{j-(k_1+k_2+\cdots+k_{i-1}+i-1)}$. It is now easy to show (using an induction on j) that there is a unique set of values for these functionals that will produce the desired values of $D^jp|_{a_j}$, and so the proof is complete.

As an example of the preceding, if we wish only to specify the values of the polynomial but not of any of its derivatives, we obtain the following.

COROLLARY 4.8. *If $x_1, x_2, ..., x_n$ are distinct points in K and $v_0, v_1, ..., v_n$ are arbitrary elements of K, then the polynomial*

$$\begin{aligned}
p=v_1&+\left[\frac{v_1}{x_1-x_2}+\frac{v_2}{x_2-x_1}\right](x-x_1)\\
&+\left[\frac{v_1}{(x_1-x_2)(x_1-x_3)}+\frac{v_2}{(x_2-x_1)(x_2-x_3)}+\frac{v_3}{(x_3-x_1)(x_3-x_2)}\right]\\
&\times(x-x_1)(x-x_2)+\cdots\\
&+\left[\frac{v_1}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}+\cdots\right.\\
&\left.+\frac{v_n}{(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})}\right](x-x_1)(x-x_2)\cdots(x-x_{n-1})
\end{aligned}$$

is the unique polynomial of degree less than or equal to $n-1$ satisfying $p(x_i)=v_i$ for $1\leqslant i\leqslant n$.

Proof. This follows easily from Corollary 4.7 and Corollary 3.11.

We are now able to prove Lemma 3.5, which completes the proof of Theorem 3.4.

Proof of Lemma 3.5. If we had a non-trivial linear relation between the $\{\varepsilon_a^j\}$, then we could write an element of P^* of the form ε_aD^j as a linear combination of other elements of P^* of this form. This is impossible, since Corollary 4.7 implies that, given any finite set of functionals of this form, we can find two polynomials on which all but one of the specified functionals agree.

We proved in Proposition 4.1 that every functional sequence has an associated polynomial sequence. If $x_1, x_2, x_3, ...$ are arbitrary elements of K, we can let $\{p_0, p_1, p_2, ...\}$ be the polynomial sequence associated with the functional sequence $\{\varepsilon_1, \varepsilon_2^2, \varepsilon_3^3, ...\}$. If we now apply Corollary 4.6, we obtain an interpolation formula that will produce the unique polynomial of degree less than or equal to n having any required values of $p(x_1)$, $p'(x_2)$, $p''(x_3)$, ..., $p^{(n)}(x_{n+1})$. We have thus proved the following.

THEOREM 4.9. *If $x_0, x_1, ..., x_n$ are arbitrary points in K and $v_0, v_1, ..., v_n$ are arbitrary elements of K, then there exists a unique polynomial p of degree less than or equal to n such that $D^i p|_{x_i} = v_i$ for $0 \leqslant i \leqslant n$.*

For example, if the points $\{x_i\}$ are evenly spaced in K, so that there are elements a, b of K such that $x_i = a + ib$, then we will show that the associated polynomial sequence $\{p_n\}$ is the sequence of *Abel polynomials*, specified by $p_0 = 1$ and $p_n = (x-a)(x-(a+nb))^{n-1}$ for $n \geqslant 1$.

LEMMA 4.10. *If $a, b \in K$, then the polynomial sequence $\{p_n\}$ defined by $p_0 = 1$ and $p_n = (x-a)(x-(a+nb))^{n-1}$ for $n \geqslant 1$ is the polynomial sequence associated with the functional sequence $\{\varepsilon_a, \varepsilon^2_{a+b}, \varepsilon^3_{a+2b}, \varepsilon^4_{a+3b}, ...\}$.*

Proof. We must show that $\varepsilon^{k+1}_{a+kb}(p_n) = \underbrace{(\varepsilon_{a+kb} \otimes \cdots \otimes \varepsilon_{a+kb})}_{k+1} \Delta^k p_n = \delta_{n,k}$. If $k > n$, then Proposition 2.7 implies that $\varepsilon^{k+1}_{a+kb}(p_n) = 0$. If $k = n$, then Proposition 2.8 implies that $\varepsilon^{k+1}_{a+kb}(p_n) = 1$. For $k < n$, we will use Proposition 2.8 and divide the sum into two parts, one consisting of the terms in which $x - a$ is among the chosen factors, and the other consisting of the terms in which it is not.

$$
\begin{aligned}
\varepsilon^{k+1}_{a+kb}(p_n) &= \underbrace{(\varepsilon_{a+kb} \otimes \cdots \varepsilon_{a+kb})}_{k+1} \Delta^k((x-a)(x-(a+nb))^{n-1}) \\
&= \binom{n-1}{k-1} ((a+kb)-(a+nb))^{n-k} \\
&\quad + \binom{n-1}{k} ((a+kb)-a)((a+kb)-(a+nb))^{n-k-1} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} b^{n-k}(k-n)^{n-k} \\
&\quad + \frac{(n-1)!}{k!(n-k-1)!} b^{n-k} k(k-n)^{n-k-1} \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!} b^{n-k}(-1)(k-n)^{n-k-1} \\
&\quad + \frac{(n-1)!}{(k-1)!(n-k-1)!} b^{n-k}(k-n)^{n-k-1} \\
&= 0.
\end{aligned}
$$

This completes the proof.

COROLLARY 4.11. *If $v_0, v_1, ..., v_n$ are arbitrary elements of K, then the polynomial*

$$p = v_0 + v_1(x-a) + \frac{v_2}{2!}(x-a)(x-(a+2b)) + \frac{v_3}{3!}(x-a)(x-(a+3b))^2$$

$$+ \frac{v_4}{4!}(x-a)(x-(a+4b))^3 + \cdots + \frac{v_n}{n!}(x-a)(x-(a+nb))^{n-1}$$

is the unique polynomial of degree less than or equal to n satisfying $D^k p|_{a+kb} = v_k$ for $0 \leqslant k \leqslant n$.

Proof. Lemma 4.10 implies that $D^k p|_{a+kb} = (k!)\,\varepsilon_{a+kb}^{k+1}(p) = (k!)(v_k/k!) = v_k$ for $0 \leqslant k \leqslant n$.

5. AUTOMORPHISMS

5.1. *Automorphisms of the Algebra P^**

In this section, we define a topology on P^* and investigate the group of *continuous* algebra automorphisms of P^*. We will show in the next section that the group of continuous algebra automorphisms of P^* is anti-isomorphic to the group of coalgebra automorphisms of P.

In Section 4.1, we defined a filtration of P^* by letting $F_n = \{f \in P^* \mid f$ can be written as a product of n elements of $P^*\}$. If I is the ideal of P^* generated by ε, then $I = P^*$ and $I^n = F_n$. Thus, if we topologize the algebra P^* by taking as basic neighborhoods of zero the sequence $\{F_n\}$, then the sequence of partial sums of the formal power series $a_0 f + a_1 f^2 + a_2 f^3 + \cdots$ converges to the element of P^* represented by that formal power series (see, e.g., [2, Chap. 10]). Thus, a continuous homomorphism $\beta\colon P^* \to P^*$ is uniquely determined by $\beta(\varepsilon)$, and we can define a continuous homomorphism $\beta\colon P^* \to P^*$ by letting $\beta(\varepsilon)$ be any element of P^* that we choose. Since every element of P^* is uniquely expressible in the form $a_0\varepsilon + a_1\varepsilon^2 + a_2\varepsilon^3 + \cdots$, the continuous homomorphism $\beta\colon P^* \to P^*$ is an *automorphism* if and only if the powers of $\beta(\varepsilon)$ form a pseudobasis to P^*, i.e., if and only if every element of P^* is uniquely expressible in the form $a_0\beta(\varepsilon) + a_1(\beta(\varepsilon))^2 + a_2(\beta(\varepsilon))^3 + \cdots$.

DEFINITION. An element f of P^* is called a *basic element* of P^* (or a *basic functional*) if $\{f, f^2, f^3, ...\}$ is a pseudobasis of P^*, i.e., if every element of P^* has a unique expression in the form $a_0 f + a_1 f^2 + a_2 f^3 + \cdots$.

The discussion at the beginning of this section proves the following.

THEOREM 5.1. *There is a one-to-one correspondence from the set of* *continuous algebra automorphisms of P^* to the set of basic elements of P^* defined by taking the image of ε.*

We will now show that $f \in P^*$ is a basic element if and only if $f(1) \neq 0$, i.e., if and only if f is not the product of two elements of P^*.

LEMMA 5.2. *If $f \in P^*$ is such that $f(1) \neq 0$, then there exists a unique polynomial sequence $\{p_0, p_1, p_2, \ldots\}$ such that $f^{n+1}(p_k) = \delta_{n,k}$.*

Proof. If $n > 0$, $f^n(x^{n-1}) = (f \otimes f \otimes \cdots \otimes f)\, \Delta^{n-1}(x^{n-1}) = (f \otimes f \otimes \cdots \otimes f)(1 \otimes 1 \otimes \cdots \otimes 1) \neq 0$, and so Corollary 3.8 implies that f^n is not of filtration $n+1$. Thus, $\{f, f^2, f^3, \ldots\}$ is a functional sequence, and the result follows from Proposition 4.1.

THEOREM 5.3. *The functional $f \in P^*$ is a basic element of P^* if and only if $f(1) \neq 0$.*

Proof. If $f(1) = 0$, then $f^n(1) = 0$ for all postive integers n. Thus, no linear combination of powers of f can equal ε, and so f is not a basic element of P^*.

Suppose now that $f(1) \neq 0$, and let $\{p_k\}$ be the polynomial sequence of Lemma 5.2. If $g \in P^*$, then $g(p_0) f + g(p_1) f^2 + g(p_2) f^3 + \cdots$ agrees with g on each element of $\{p_k\}$. Since $\{p_k\}$ is a basis of P, this implies that $g = g(p_0) f + g(p_1) f^2 + g(p_2) f^3 + \cdots$. If $\{a_0, a_1, a_2, \ldots\}$ is such that $g = a_0 f + a_1 f^2 + a_2 f^3 + \cdots$, then, since $f^{n+1}(p_k) = \delta_{n,k}$, we must have $a_k = g(p_k)$ for all k, and the proof is complete.

DEFINITION. We will call the polynomial sequence of Lemma 5.2 the polynomial sequence *associated* with the basic functional f.

COROLLARY 5.4. *If f is a basic element of P^* and g is any element of P^*, then there exists a unique continuous homomorphism $\beta\colon P^* \to P^*$ such that $\beta(f) = g$.*

Proof. Since every element of P^* can be written in a unique way as a power series in f, the result is clear.

5.2. *Automorphisms of P and of P^**

If $\alpha\colon P \to P$ is a homomorphism of the coalgebra P, then α^*, the dual of α, is a continuous homomorphism $\alpha^*\colon P^* \to P^*$. If α is an automorphism, then $(\alpha^{-1})^* = (\alpha^*)^{-1}$, and so α^* is a continuous automorphism of the algebra P^*. In this section, we show that this in fact defines an anti-isomorphism from the group of coalgebra automorphisms of P to the group of continuous algebra automorphisms of P^*.

THEOREM 5.5. *The operation of taking the dual defines an anti-isomorphism from the group of coalgebra automorphisms of P to the group of continuous algebra automorphisms of P^*.*

Proof. If $\alpha: P \to P$ is a coalgebra automorphism, then α^*, the dual of α, is a continuous homomorphism $P^* \to P^*$. Since α is an automorphism, α^{-1} exists, and so $(\alpha^{-1})^* = (\alpha^*)^{-1}$. Thus, α^* is a continuous algebra automorphism. Since $(\alpha_1 \alpha_2)^* = \alpha_2^* \alpha_1^*$, this defines an anti-homomorphism from the group of coalgebra automorphisms of P to the group of continuous algebra automorphisms of P^*. We will show that this is a bijection by constructing an inverse function.

If β: $P^* \to P^*$ is a continuous algebra automorphism, then β^* is a linear automorphism of P^{**}. We will now show that under the natural embedding of P as a subspace of P^{**}, the dual of the algebra multiplication of P^* induces the comultiplication of P and β^* induces an automorphism of the coalgebra P.

We define $i: P \to P^{**}$ by letting $(i(p))(f) = f(p)$ for $f \in P^*$. This is an embedding because if $i(p) = 0$, then $(i(p))(f) = 0$ for all $f \in P^*$. In particular, $(i(p))(\varepsilon^n) = \varepsilon^n(p) = 0$ for all n, and so $p = 0$. We will let $\hat{P}$ denote the image of P in P^{**}, and if $p \in P$, we will let $\hat{p}$ denote $i(p)$. *Assertion*: $\hat{P} = \{h \in P^{**} \mid \text{For some integer } n,\ h(a_0\varepsilon + a_1\varepsilon^2 + \cdots) = h(a_0\varepsilon + a_1\varepsilon^2 + \cdots + a_n\varepsilon^{n+1})\}$. This is because if $p \in P$ is of dimension n, then $\hat{p}(a_0\varepsilon + a_1\varepsilon^2 + \cdots) = \hat{p}(a_0\varepsilon + a_1\varepsilon^2 + \cdots + a_n\varepsilon^{n+1})$, since $\varepsilon^{n+2}(p) = 0$. Conversely, if $h \in P^{**}$ is such that $h(a_0\varepsilon + a_1\varepsilon^2 + \cdots) = h(a_0\varepsilon + a_1\varepsilon^2 + \cdots + a_n\varepsilon^{n+1})$, then we can define $p \in P$ by letting $p = h(\varepsilon^{n+1})\,x^n + h(\varepsilon^n)\,x^{n-1} + \cdots + h(\varepsilon)$; then $\hat{p}(a_0\varepsilon + a_1\varepsilon^2 + \cdots) = (a_0\varepsilon + a_1\varepsilon^2 + \cdots)(p) = (a_0\varepsilon + a_1\varepsilon^2 + \cdots)(h(\varepsilon^{n+1})\,x^n + h(\varepsilon^n)\,x^{n-1} + \cdots + h(\varepsilon)) = a_0 h(\varepsilon) + a_1 h(\varepsilon^2) + \cdots + a_n h(\varepsilon^{n+1}) = h(a_0\varepsilon + a_1\varepsilon^2 + \cdots + a_n\varepsilon^{n+1}) = h(a_0\varepsilon + a_1\varepsilon^2 + \cdots)$. Thus, $\hat{p} = h$, and the assertion is proved.

We will now show that β^*: $P^{**} \to P^{**}$ carries $\hat{P}$ into itself. Let $f = \beta(\varepsilon)$, and let $\hat{p} \in \hat{P}$ be of dimension n; then $(\beta^*(\hat{p}))(a_0\varepsilon + a_1\varepsilon^2 + \cdots) = \hat{p}(\beta(a_0\varepsilon + a_1\varepsilon^2 + \cdots)) = \hat{p}(a_0 f + a_1 f^2 + \cdots) = (a_0 f + a_1 f^2 + \cdots)(p) = (a_0 f + a_1 f^2 + \cdots + a_n f^{n+1})(p) = (\beta(a_0\varepsilon + a_1\varepsilon^2 + \cdots + a_n\varepsilon^{n+1}))(p) = \hat{p}(\beta(a_0\varepsilon + a_1\varepsilon^2 + \cdots + a_n\varepsilon^{n+1})) = (\beta^*(\hat{p}))(a_0\varepsilon + a_1\varepsilon^2 + \cdots + a_n\varepsilon^{n+1})$. Thus, $\beta^*(\hat{p}) \in \hat{P}$. Since $(\beta^{-1})^* = (\beta^*)^{-1}$, the restriction of β^* to $\hat{P}$ is a linear automorphism of $\hat{P}$.

We will now show that the dual of the multiplication in P^*, when restricted to $\hat{P}$, induces a comultiplication in $\hat{P}$ that corresponds (under the linear isomorphism $i: P \cong \hat{P}$) to the comultiplication Δ. Since the multiplication in P^* equals $\Delta^*: P^* \otimes P^* \to P^*$, the dual of the multiplication in P^* is $\Delta^{**}: P^{**} \to (P^* \otimes P^*)^*$. If $f, g \in P^*$, then

$$\begin{aligned}
\Delta^{**}(\widehat{x^n})(f \otimes g) &= \widehat{x^n}\,(\Delta^*(f \otimes g)) \\
&= (\Delta^*(f \otimes g))(x^n) \\
&= (f \otimes g)(\Delta(x^n)) \\
&= \sum_{i=0}^{n-1} f(x^i)\, g(x^{n-1-i}) \\
&= \sum_{i=0}^{n-1} \widehat{x^i}(f)\, \widehat{x^{n-1-i}}(g) \\
&= \left(\sum_{i=0}^{n-1} \widehat{x^i} \otimes \widehat{x^{n-1-i}}\right)(f \otimes g).
\end{aligned}$$

Thus, $\Delta^{**}(\widehat{x^n}) = \sum_{i=0}^{n-1} \widehat{x^i} \otimes \widehat{x^{n-1-i}}$, which is an element of the subspace $\hat{P} \otimes \hat{P}$ of the subspace $P^{**} \otimes P^{**}$ of $(P^* \otimes P^*)^*$. It we now let $\hat{\Delta}$ be the restriction of Δ^{**} to $\hat{P}$, then we have a comultiplication $\hat{\Delta}: \hat{P} \to \hat{P} \otimes \hat{P}$, and we have proved that $\hat{\Delta}(i(x^n)) = (i \otimes i)\, \Delta(x^n)$. Since $\{x^n\}$ is a basis of P, this implies that $\hat{\Delta} i = (i \otimes i) \Delta$ on all of P, and so the linear isomorphism $i: P \to \hat{P}$ is an isomorphism of coalgebras.

We have proved that $P \cong \hat{P}$ as coalgebras, and that a continuous algebra automorphism β of P^* induces a coalgebra automorphism β^* of $\hat{P} \cong P$. It is straightforward to check that if $\alpha: P \to P$ is a coalgebra automorphism, then $\alpha^{**} = \alpha$ and if $\beta: P^* \to P^*$ is a continuous algebra automorphism, then $\beta^{**} = \beta$, and so the proof is complete.

5.3. *Automorphisms of the Coalgebra P*

The coalgebra P could have been defined as the vector space with basis $\{1, x, x^2, \ldots\}$ and with comultiplication defined by $\Delta(x^n) = \sum_{i=0}^{n-1} x^i \otimes x^{n-1-i}$. In this section, we define a *Newtonian sequence* to be a polynomial sequence that could have been used to define P in this way, and we show that every automorphism of the coalgebra P is obtained by sending the sequence $\{1, x, x^2, \ldots\}$ to the corresponding elements of some Newtonian sequence. We will obtain a classification of Newtonian sequences in Section 7.1.

DEFINITION. An *automorphism* of the coalgebra P is a linear isomorphism $\alpha: P \to P$ such that $\Delta\alpha = (\alpha \otimes \alpha)\Delta$.

Let $\alpha: P \to P$ be an automorphism of the coalgebra P. If we let $p_n = \alpha(x^n)$, then the sequence $\{p_n\}$ satisfies $\Delta p_n = \sum_{i=1}^{n-1} p_i \otimes p_{n-1-i}$.

DEFINITION. A *Newtonian sequence* is a polynomial sequence $\{p_n\}$ such that $\Delta(p_n) = \sum_{i=0}^{n-1} p_i \otimes p_{n-1-i}$.

EXAMPLE. If $a \in K$, then Proposition 2.8 implies that the sequence $\{(x-a)^n\}_{n=0}^{\infty}$ is a Newtonian sequence.

THEOREM 5.6. *There is a one-to-one correspondence between the set of coalgebra automorphisms of P and the set of Newtonian sequences. If $\alpha: P \to P$ is a coalgebra automorphism, then the corresponding Newtonian sequence is given by $p_n = \alpha(x^n)$.*

Proof. If $\alpha: P \to P$ is a coalgebra automorphism, then $\Delta\alpha(x^n) = (\alpha \otimes \alpha)\Delta(x^n)$, i.e., $\Delta p_n = \sum_{i=0}^{n-1} p_i \otimes p_{n-1-i}$. Since α is an automorphism, $p_0 \neq 0$, and since $\Delta(1) = 0$, we must have $p_0 \in \text{kernel}(\Delta)$. Proposition 2.6 now implies that p_0 must be a non-zero constant. Since $\Delta p_1 = p_0 \otimes p_0$, the polynomial p_1 must be of degree 1. It is now a straightforward induction argument to show that p_n is of degree n. Thus, $\{p_n\}$ is a Newtonian sequence.

To complete the proof, we will show that each Newtonian sequence arises in this way from a unique coalgebra automorphism. Let $\{p_n\}$ be a Newtonian sequence. Since $\{x^n\}$ and $\{p_n\}$ are bases of P, the map $\alpha: P \to P$ defined by $\alpha(x^n) = p_n$ is a linear automorphism. Since

$$\begin{aligned}\Delta\alpha(x^n) &= \Delta p_n \\ &= \sum_{i=0}^{n-1} p_i \otimes p_{n-1-i} \\ &= (\alpha \otimes \alpha) \sum_{i=0}^{n-1} x^i \otimes x^{n-1-i} \\ &= (\alpha \otimes \alpha)\Delta(x^n),\end{aligned}$$

the map α is an automorphism of the coalgebra P, and the proof is complete.

THEOREM 5.7. *If f is a basic functional, then the associated polynomial sequence $\{p_n\}$ is a Newtonian sequence. This defines a one-to-one correspondence between the set of basic functionals and the set of Newtonian sequences.*

Proof. If f is a basic functional, let ϕ_f denote the continuous automorphism of P^* defined by $\phi_f(\varepsilon)=f$. Since $\phi_f^{-1}\phi_f=$ the identity of P^*, we have

$$\begin{aligned}\langle \varepsilon^{k+1} \mid x^n\rangle &= \langle \phi_f^{-1}\phi_f\varepsilon^{k+1} \mid x^n\rangle \\ &= \langle \phi_f\varepsilon^{k+1} \mid (\phi_f^{-1})^* x^n\rangle \\ &= \langle f^{k+1} \mid (\phi_f^{-1})^* x^n\rangle.\end{aligned}$$

If we let $p_n=(\phi_f^{-1})^*(x^n)$, then we have $f^{k+1}(p_n)=\varepsilon^{k+1}(x^n)=\delta_{k,n}$, and so $\{p_n\}$ is the polynomial sequence associated with f. Since $(\phi_f^{-1})^*$ is an automorphism of the coalgebra P and $p_n=(\phi_f^{-1})^*(x^n)$, Theorem 5.6 implies that $\{p_n\}$ is a Newtonian sequence.

If $\{p_n\}$ is a Newtonian sequence, we can define an automorphism α of the coalgebra P by letting $\alpha(x^n)=p_n$. We then have the continuous automorphism $(\alpha^{-1})^*$ of P^*, and if we let $f=(\alpha^{-1})^*(\varepsilon)$, then f is a basic functional and $\phi_f=(\alpha^{-1})^*$. Thus, $\alpha=(\phi_f^{-1})^*$, and so $\{p_n\}$ is the polynomial sequence associated with f.

Theorem 5.7 allows us to characterize Newtonian sequences in terms of their associated functional sequences.

COROLLARY 5.8. *Let $\{p_n\}$ be a polynomial sequence with associated functional sequence $\{f_1, f_2, f_3, ...\}$. The polynomial sequence $\{p_n\}$ is a Newtonian sequence if and only if $f_n=f_1^n$ for all $n \geqslant 1$.*

DEFINITION. If f is a basic functional, we will call its associated polynomial sequence the Newtonian sequence *associated* with f.

The proof of Theorem 5.7 also proves the following.

PROPOSITION 5.9. *If f is a basic functional with associated Newtonian sequence $\{p_n\}$, and if $\phi_f\colon P^* \to P^*$ is defined by $\phi_f(\varepsilon)=f$, then $\phi_f^*(p_n)=x^n$, i.e., $(\phi_f^{-1})^*(x^n)=p_n$.*

We are now able to show that there is really nothing special about the basic functional ε and its associated Newtonian sequence $\{x^n\}$; the algebra structure of P^* can just as easily be described using any other basic functional.

PROPOSITION 5.10. *If f is a basic functional with associated Newtonian sequence $\{p_n\}$, then f determines a continuous isomorphism of algebras $\psi_f\colon P^* \cong FPS_0$ given by $\psi_f(g)=g(p_0)t+g(p_1)t^2+g(p_2)t^3+\cdots$.*

Proof. The proof of Theorem 3.7 works here as well, where we replace the Newtonian sequence $\{x^n\}$ with the Newtonian sequence $\{p_n\}$.

If f is basic functional with associated Newtonian sequence $\{p_n\}$, then the sequence of powers of f $\{f, f^2, f^3, \ldots\}$ is the functional sequence associated with the polynomial sequence $\{p_n\}$. Theorem 4.3 now implies the following.

THEOREM (Expansion Theorem for Newtonian Sequences) 5.11. *Let f be a basic functional with associated Newtonian sequence $\{p_n\}$. If L is either an element of P^* or a linear operator on P, then*

$$L = \sum_{n=0}^{\infty} L(p_n)\, f^{n+1}.$$

As an example of the preceding, if $a \in K$ and we use the polynomial sequence $p_n = (x-a)^n$ (which Lemma 4.4 shows to be the Newtonian sequence associated with the basic functional ε_a), we obtain Taylor's Theorem.

COROLLARY (Taylor's Theorem) 5.12. *If $p \in P$, then $p = \sum_{k=0}^{\infty} (1/k!)(x-a)^k \varepsilon_a D^k p$.*

Proof. Applying Theorem 5.11 to the identity operator yields $\mathrm{Id} = \sum_{k=0}^{\infty} (x-a)^k \varepsilon_a^{k+1} = \sum_{k=0}^{\infty} (1/k!)(x-a)^k \varepsilon_a D^k$. The result follows by applying this to p.

6. ADJOINT OPERATORS

6.1. *Strictly Descending Operators*

In this section, we use our comultiplication Δ to define operators on P that are the products of elements of P^* and operators on P. We use this product to define the adjoints of operators on P^* that are defined as multiplication by a fixed functional.

Notation. We will use $\mathrm{Op}(P)$ to denote the algebra of linear operators on P.

DEFINITION. Let $L \in \mathrm{Op}(P)$ and let $f \in P^*$. The product Lf is defined to be the operator $(L \otimes f)\Delta$, and the product fL is defined to be the operator $(f \otimes L)\Delta$.

Remark 6.1. Strictly speaking, $(L \otimes f)\Delta$ is a function from P to $P \otimes K$, but we are implicitly passing through the natural isomorphism between $P \otimes K$ and P. We shall often use this natural isomorphism without explicitly mentioning that we are doing so.

Since we proved in Proposition 2.3 that Δ is cocommutative, we have $Lf = fL$. Since we proved in Proposition 2.2 that Δ is coassociative, if $f_1, f_2, ..., f_n \in P^*$, then $Lf_1 f_2 \cdots f_n = f_1 L f_2 \cdots f_n = f_1 f_2 L f_3 \cdots f_n = \cdots = f_1 f_2 \cdots f_n L$, where the products of $n+1$ factors can be associated arbitrarily.

EXAMPLE 6.2. Let ε_x denote the identity operator on P. If $n > 0$, then

$$(\varepsilon\varepsilon_x)(x^n) = \sum_{i=0}^{n-1} \varepsilon(x^i)\, x^{n-1-i} = \begin{cases} x^{n-1}, & \text{if } n \geqslant 1 \\ 0, & \text{if } n = 1. \end{cases}$$

In general,

$$(\varepsilon^k \varepsilon_x)\, x^n = \sum_{i=0}^{n-1} \varepsilon^k(x^i)\, x^{n-1-i} = \begin{cases} x^{n-k}, & \text{if } n \geqslant k; \\ 0, & \text{if } n < k. \end{cases}$$

More generally, if f is any basic functional and $\{p_n\}$ is its associated Newtonian sequence, then

$$(f^k \varepsilon_x)\, p_n = \sum_{i=0}^{n-1} f^k(p_i)\, p_{n-1-i} = \begin{cases} p_{n-k}, & \text{if } n \geqslant k; \\ 0, & \text{if } n < k. \end{cases}$$

For any $f \in P^*$, multiplication with f defines a linear operator on P^*. We will now show that the adjoint operator on P is $f\varepsilon_x$.

PROPOSITION 6.3. *If f, $g \in P^*$ and $p \in P$, then*

$$\langle fg \mid p \rangle = \langle g \mid f\varepsilon_x(p) \rangle.$$

Proof.

$$\begin{aligned} \langle fg \mid p \rangle &= (f \otimes g)\, \Delta p \\ &= (1 \otimes g)(f \otimes \varepsilon_x)\, \Delta p \\ &= (1 \otimes g)(f\varepsilon_x(p)) \\ &= \langle g \mid f\varepsilon_x(p) \rangle. \end{aligned}$$

Notation. If $f \in P^*$, we will use f^* to denote $f\varepsilon_x$.

DEFINITION. An operator on P of the form f^* for some $f \in P^*$ will be called a *strictly descending operator*. A strictly descending operator f^* for f a basic functional will be called a *basic operator*.

Example 6.2 shows that a basic operator f^* is a sort of "generalized differentiation" relative to the Newtonian sequence associated with f, in the same way that ε^* is a differentiation relative to the Newtonian sequence $\{x^n\}$. In fact, if ϕ_f is the automorphism of P^* that takes ε to f, then Proposition 5.9 implies that $f^* = (\phi_f^{-1})^* \varepsilon^* \phi_f^*$.

The following are easy consequences of the definitions.

PROPOSITION 6.4. *If $f \in P^*$ is given by $f = a_0\varepsilon + a_1\varepsilon^2 + a_2\varepsilon^3 + \cdots$, then*

$$f^*(x^n) = a_0 x^{n-1} + a_1 x^{n-2} + a_2 x^{n-3} + \cdots + a_{n-1}.$$

COROLLARY 6.5. *The strictly descending operator f^* is a basic operator if and only if there is some $n > 0$ such that $f^*(x^n)$ is of degree exactly $n-1$.*

COROLLARY 6.6. *The strictly descending operator f^* is a basic operator if and only if $f^*(x) \neq 0$.*

6.2. *The Lagrange Interpolation Formula*

As an application, we prove the Lagrange interpolation formula.

THEOREM (Lagrange Interpolation Formula) 6.7. *If $x_1, x_2, ..., x_n$ are distinct elements of K then, as operators on P,*

$$\begin{aligned} \mathrm{Id} = {} & \varepsilon_1 \frac{(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} + \varepsilon_2 \frac{(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} \\ & + \cdots + \varepsilon_n \frac{(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} + R_n, \end{aligned}$$

where $R_n = (x-x_1)(x-x_2)\cdots(x-x_n)\,\varepsilon_1^*\varepsilon_2^*\varepsilon_3^*\cdots\varepsilon_n^*$.

Proof. Since

$$(\varepsilon_x \varepsilon_a)\, x^n = \sum_{i=0}^{n-1} x^i a^{n-1-i} = \frac{x^n - a^n}{x-a},$$

we have $\varepsilon_x\varepsilon_a = \varepsilon_x/(x-a) + \varepsilon_a/(a-x)$. Using this relation along with $\varepsilon_a\varepsilon_b = \varepsilon_a/(a-b) + \varepsilon_b/(b-a)$ for $a \neq b$, it is straightforward induction to show that there are elements $a_x, a_1, a_2, ..., a_n$ of the quotient field of P such that

$$a_x\varepsilon_x + a_1\varepsilon_1 + \cdots + a_n\varepsilon_n = \varepsilon_x\varepsilon_1\varepsilon_2\cdots\varepsilon_n$$

as linear functions from P to the quotient field of P. We can now repeat the arguments in the proofs of Theorem 3.10 and Corollary 3.11 to show that

$$\frac{\varepsilon_x}{\prod_{i=1}^{n}(x-x_i)}+\frac{\varepsilon_1}{(x_1-x)(\prod_{i\neq 1}(x_1-x_i))}+\frac{\varepsilon_2}{(x_2-x)(\prod_{i\neq 2}(x_2-x_i))}$$

$$+\cdots+\frac{\varepsilon_n}{(x_n-x)(\prod_{i\neq n}(x_n-x_i))}=\varepsilon_x\varepsilon_1\varepsilon_2\cdots\varepsilon_n.$$

It we now multiply both sides of this equation by $\prod_{i=1}^{n}(x-x_i)$ and solve for $\varepsilon_x=\mathrm{Id}$, we obtain the equation that we require.

6.3. *The Algebra of Descending Operators*

In this section, we define a descending operator to be an operator on P that is the adjoint of an operator on P^* defined by multiplication by an element of $K[[\varepsilon]]$, the algebra of formal power series in the variable ε. The descending operators form a subalgebra of $\mathrm{Op}(P)$ which contains the strictly descending operators as a subalgebra. We will also show that the algebra of strictly desncending operators is isomorphic (as a topological algebra) to P^*, and that the algebra of descending operators is exactly the algebra of comodule maps $P\to P$ (where we consider P as a comodule over itself).

Let $K[[\varepsilon]]$ be the algebra of formal power series in the variable ε. Theorem 3.7 allows us to look upon the algebra P^* as the augmentation ideal of $K[[\varepsilon]]$, i.e., as the formal power series with zero constant term. Thus, if $M\in K[[\varepsilon]]$ and $f\in P^*$, the product $Mf\in P^*$. This defines a linear operator (which we will also call M) on P^*. We will now compute the adjoint of the operator M.

DEFINITION. If $M\in K[[\varepsilon]]$ is given by $M=a_0+a_1\varepsilon+a_2\varepsilon^2+\cdots$, let M^* be the operator on P defined by

$$M^*(x^n)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n.$$

PROPOSITION 6.8. *If $f\in P^*$, $p\in P$, and $M\in K[[\varepsilon]]$, then*

$$\langle Mf\mid p\rangle=\langle f\mid M^*p\rangle.$$

Proof. It is sufficient to verify this in the case $p=x^n$. If $M=a_0+a_1\varepsilon+a_2\varepsilon^2+\cdots$ and we let $g=a_1\varepsilon+a_2\varepsilon^2+a_3\varepsilon^3+\cdots$, then the operator M on P is given by $Mf=a_0f+fg$. Thus,

$$\begin{aligned}\langle Mf \mid x^n\rangle &= \langle a_0 f + fg \mid x^n\rangle \\ &= \langle a_0 f \mid x^n\rangle + \langle fg \mid x^n\rangle \\ &= \langle f \mid a_0 x^n\rangle + \langle f \mid g^* x^n\rangle \\ &= \langle f \mid a_0 x^n\rangle + \langle f \mid a_1 x^{n-1} + a_2 x^{n-2} + a_3 x^{n-3} + \cdots + a_n\rangle \\ &= \langle f \mid a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n\rangle \\ &= \langle f \mid M^* x^n\rangle.\end{aligned}$$

The next proposition shows that the above construction actually embeds $K[[\varepsilon]]$ as a subalgebra of $\mathrm{Op}(P)$.

PROPOSITION 6.9. *If $M, N \in K[[\varepsilon]]$, then $M^*N^* = (MN)^*$.*

Proof. Since $\{x^n\}$ is a basis of P and $\{\varepsilon, \varepsilon^2, \varepsilon^3, ...\}$ is a pseudobasis of P^*, it is sufficient to show that $\langle \varepsilon^k \mid M^*N^*(x^n)\rangle = \langle \varepsilon^k \mid (MN)^*(x^n)\rangle$ for all $k \geqslant 1$ and $n \geqslant 0$. Since

$$\begin{aligned}\langle \varepsilon^k \mid (MN)^*(x^n)\rangle &= \langle \varepsilon^k MN \mid x^n\rangle \\ &= \langle \varepsilon^k M \mid N^*(x^n)\rangle \\ &= \langle \varepsilon^k \mid M^*N^*(x^n)\rangle,\end{aligned}$$

the proof is complete.

DEFINITION. A *descending operator* is a linear operator on P of the form M^* for some $M \in K[[\varepsilon]]$.

The algebra $\mathrm{Op}(P)$ of linear operators on P has a natural topology, which we shall now describe. We define a filtration $\mathrm{Op}(P) = F_0 \supset F_1 \supset F_2 \supset \cdots$ of $\mathrm{Op}(P)$ by letting $F_n = \{L \in \mathrm{Op}(P) \mid$ the kernel of L contains all polynomials of degree less than $n\}$. The topology on $\mathrm{Op}(P)$ is the one obtained by taking the F_n as basic neighborhoods of zero (see, e.g., [2, Chap. 10]).

The following is now easy to verify.

PROPOSITION 6.10. *If we topologize $K[[\varepsilon]]$ by taking as basic neighborhoods of zero the powers of the ideal generated by ε, then the above construction defines an isomorphism of topological algebras between $K[[\varepsilon]]$ and the algebra of descending operators.*

We will now show that the *invertible* descending operators are exactly those that are not strictly descending operators.

PROPOSITION 6.11. *If M^* is a descending operator, then the following are equivalent*:

(1) *M^* is invertible.*

(2) *$M^*(x^n)$ is of degree n for some $n \geqslant 0$.*

(3) *$M^*(x^n)$ is of degree n for every $n \geqslant 0$.*

(4) *$M^*(1) \neq 0$.*

(5) *If $M = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$, then $a_0 \neq 0$.*

(6) *M^* is not a strictly descending operator.*

Proof. (5) *implies* (1). Since $a_0 \neq 0$, M is invertible in $K[[\varepsilon]]$. Proposition 6.9 now implies that $(M^{-1})^* = (M^*)^{-1}$.

(1) *implies* (5). If $M = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$ and $a_0 = 0$, then the kernel of M^* contains the constant polynomials, and so M^* cannot be invertible.

The definition of M^* easily implies that (5) is equivalent to each of (2), (3), (4), and (6), and so the proof is complete.

COROLLARY 6.12. *If M^* is a descending operator, then M^* is invertible in* Op(P) *if and only if it is invertible in the algebra of descending operators.*

We can now characterize the elements of Op(P) that are descending operators.

THEOREM 6.13. *If $L \in \mathrm{Op}(P)$, then the following are equivalent*:

(1) *L is a descending operator.*

(2) *$L\varepsilon^* = \varepsilon^* L$.*

(3) *$Lf^* = f^* L$ for some basic functional f.*

(4) *$\Delta L = (L \otimes 1)\Delta$.*

(5) *$\Delta L = (1 \otimes L)\Delta$.*

Proof. (1) *implies* (2). Let $L = M^*$ for $M = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$; then

$$\begin{aligned} L\varepsilon^*(x^n) &= L(x^{n-1}) \\ &= a_0 x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-1} \\ &= \varepsilon^*(a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n) \\ &= \varepsilon^* L(x^n). \end{aligned}$$

(2) *implies* (1). For every $k \geqslant 0$ let $a_k = \langle \varepsilon \mid L(x^k) \rangle$, and let $M = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$; we will show that $L = M^*$.

Since $\{x^n\}$ is a basis of P and $\{\varepsilon, \varepsilon^2, \varepsilon^3, \ldots\}$ is a pseudobasis of P^*, it is sufficient to show that $\langle \varepsilon^k \mid L(x^n) \rangle = \langle \varepsilon^k \mid M^*(x^n) \rangle$ for all $n \geqslant 0$ and $k \geqslant 1$. If $k > n+1$, then

$$\begin{aligned}
\langle \varepsilon^k \mid L(x^n) \rangle &= \langle \varepsilon \mid (\varepsilon^*)^{k-1} L(x^n) \rangle \\
&= \langle \varepsilon \mid L(\varepsilon^*)^{k-1}(x^n) \rangle \\
&= \langle \varepsilon \mid 0 \rangle \\
&= 0 \\
&= \langle \varepsilon^k \mid M^*(x^n) \rangle.
\end{aligned}$$

If $k \leqslant n+1$, then

$$\begin{aligned}
\langle \varepsilon^k \mid L(x^n) \rangle &= \langle \varepsilon \mid (\varepsilon^*)^{k-1} L(x^n)) \\
&= \langle \varepsilon \mid L(\varepsilon^*)^{k-1}(x^n) \rangle \\
&= \langle \varepsilon \mid L(x^{n-k+1}) \rangle \\
&= a_{n-k+1} \\
&= \langle \varepsilon^k \mid a_{n-k+1} x^{k-1} \rangle \\
&= \langle \varepsilon^k \mid a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n x^0 \rangle \\
&= \langle \varepsilon^k \mid M^*(x^n) \rangle.
\end{aligned}$$

(2) *implies* (3). This is obvious.

(3) *implies* (2). Since f is a basic functional, we can write ε in the form $\varepsilon = a_0 f + a_1 f^2 + a_2 f^2 + \cdots$. The result now follows easily.

(1) *implies* (4). Let $L = M^*$ for $M = a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + \cdots$. If we let $f = a_1 \varepsilon + a_2 \varepsilon^2 + a_3 \varepsilon^3 + \cdots$, then $L(p) = a_0 p + f^*(p) = a_0 p + (f \otimes 1)\,\Delta(p)$. Thus,

$$\begin{aligned}
\Delta L(p) &= \Delta(a_0 p + (f \otimes\ )\,\Delta p) \\
&= (a_0 \otimes 1)\,\Delta p + \Delta(f \otimes 1)\,\Delta p \\
&= (a_0 \otimes 1)\,\Delta p + (1 + \Delta)(f \otimes 1)\,\Delta p \qquad \text{(using Remark 6.1)} \\
&= (a_0 \otimes 1)\,\Delta_p + (f \otimes 1 \otimes 1)(1 \otimes \Delta)\,\Delta p \\
&= (a_0 \otimes 1)\,\Delta p + (f \otimes 1 \otimes 1)(\Delta \otimes 1)\,\Delta p \qquad \text{(using Proposition 2.2)} \\
&= (a_0 \otimes 1)\,\Delta p + (((f \otimes 1)\Delta) \otimes 1)\,\Delta p \\
&= (a_0 \otimes 1)\,\Delta p + (f^* \otimes 1)\,\Delta p \\
&= (L \otimes 1)\,\Delta p.
\end{aligned}$$

(4) *implies* (2).

$$\begin{aligned}\varepsilon^* L &= (1 \otimes \varepsilon)\, \Delta L \\ &= (1 \otimes \varepsilon)(L \otimes 1)\Delta \\ &= (L \otimes \varepsilon)\Delta \\ &= (L \otimes 1)(1 \otimes \varepsilon)\Delta \\ &= L\varepsilon^*.\end{aligned}$$

(4) *implies* (5). If $T\colon P \otimes P \to P \otimes P$ is defined by $T(a \otimes b) = b \otimes a$, then

$$\begin{aligned}(1 \otimes L)\Delta &= T(L + 1)\, T\Delta \\ &= T(L \otimes 1)\Delta \qquad \text{(using Proposition 2.3)} \\ &= T\, \Delta L \\ &= \Delta L \qquad \text{(using Proposition 2.3 again)}.\end{aligned}$$

(5) *implies* (4). This is similar to (4) implies (5).

Composing our identification of P^* with the augmentation ideal of $K[[\varepsilon]]$ with the embedding of $K[[\varepsilon]]$ in $\mathrm{Op}(P)$, we obtain the following.

THEOREM 6.14. *The algebra of strictly descending operators is isomorphic as a topological algebra to P^*.*

We have defined the algebra of descending operators in terms of the basic functional ε. We will now show that this was not essential, and that any basic functional f would produce the same operators.

Notation. If $M \in K[[\varepsilon]]$ and f is a basic functional, we will use M_f^* to denote the operator $(\phi_f^*)^{-1} M^* \phi_f^*$ on P (where ϕ_f is the continuous automorphism of P^* that takes ε to f).

PROPOSITION 6.15. *If f is a basic functional and $M = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$, then $M_f^* = a_0 + a_1 f^* + a_2(f^*)^2 + \cdots$.*

Proof. Let $\{p_n\}$ be the Newtonian sequence associated with f. Since $\{p_n\}$ is a basis of P, it is sufficient to show that $M_f^*(p_n) = a_0 p_n + a_1 f^*(p_n) + a_2(f^*)^2(p_n) + \cdots$ for all $n \geq 0$. Since

$$\begin{aligned} M_f^*(p_n) &= (\phi_f^*)^{-1} M^* \phi_f^*(p_n) \\ &= (\phi_f^*)^{-1} M^*(x^n) \qquad \text{(using Proposition 5.9)} \\ &= (\phi_f^*)^{-1}(a_0 x^n + a_1 x^{n-1} + \cdots + a_n) \\ &= a_0 p_n + a_1 p_{n-1} + \cdots + a_n p_0 \\ &= a_0 p_n + a_1 f^*(p_n) + a_2 (f^*)^2 (p_n) + \cdots \qquad \text{(using Example 6.2)}, \end{aligned}$$

the proof is complete.

COROLLARY 6.16. *If f is a basic functional with associated Newtonian sequence $\{p_n\}$, then*

$$M_f^*(p_n) = a_0 p_n + a_1 p_{n-1} + a_2 p_{n-2} + \cdots + a_n p_0.$$

Proof. This follows from Proposition 6.15 and Example 6.2.

COROLLARY 6.17. *If $M \in K[[\varepsilon]]$ and f is a basic functional, then M_f^* is a descending operator.*

Proof. Since ϕ_f^* is a coalgebra map, we have

$$\begin{aligned} \Delta M_f^* &= \Delta (\phi_f^*)^{-1} M^* \phi_f^* \\ &= ((\phi_f^*)^{-1} \otimes (\phi_f^*)^{-1}) \Delta M^* \phi_f^* \\ &= ((\phi_f^*)^{-1} \otimes (\phi_f^*)^{-1})(M^* \otimes 1) \Delta \phi_f^* \qquad \text{(using Theorem 6.13)} \\ &= ((\phi_f^*)^{-1} \otimes (\phi_f^*)^{-1})(M^* \otimes 1)(\phi_f^* \otimes \phi_f^*) \Delta \\ &= (((\phi_f^*)^{-1} M^* \phi_f^*) \otimes 1) \Delta \\ &= (M_f^* \otimes 1) \Delta, \end{aligned}$$

and the result follows from Theorem 6.13.

COROLLARY 6.18. *If f is a basic functional, then the function that takes the descending operator M^* to the descending operator M_f^* is an automorphism of the algebra of descending operators.*

COROLLARY 6.19. *If M^* is a descending operator and f is a basic functional, then M_f^* is invertible if and only if M^* is invertible.*

Proof. This follows from Corollary 6.12.

Corollary 6.18 implies that every descending operator can be expanded in terms of an arbitrary basic operator f^*. The next theorem shows how to do this.

THEOREM (Expansion Theorem for Descending Operators) 6.20. *Let f be a basic functional with associated Newtonian sequence $\{p_n\}$. If L is a descending operator, then*

$$L = \sum_{k=0}^{\infty} \langle f \mid Lp_k \rangle (f^*)^k.$$

Proof. Corollary 6.18 implies that the descending operator L can be written in the form

$$L = a_0 + a_1 f^* + a_2 (f^*)^2 + \cdots.$$

Since

$$Lp_k = a_0 p_k + a_1 p_{k-1} + a_2 p_{k-2} + \cdots + a_k p_0,$$

we have

$$\begin{aligned} \langle f \mid Lp_k \rangle &= a_0 \langle f \mid p_k \rangle + a_1 \langle f \mid p_{k-1} \rangle + \cdots + a_k \langle f \mid p_0 \rangle \\ &= a_k, \end{aligned}$$

and the proof is complete.

Remark. We have defined the algebra of descending operators to be the algebra of adjoints of a certain subalgebra of $\mathrm{Op}(P^*)$. We have also shown that the appropriate subalgebra of $\mathrm{Op}(P^*)$ can be defined by using any basic functional to identify P^* with the augmentation ideal of $K[[t]]$. In fact, this subalgebra of $\mathrm{Op}(P^*)$ is just the direct sum of the operators that multiply by a fixed scalar and the operators that multiply by a fixed functional, and there was really no need to choose a basic functional (except that it allows one to write down convenient formulas for these operators). We will now show that the algebra of descending operators can be defined without making reference to P^* at all.

We will now define the notion of a *comodule over a coalgebra*, which is dual to the notion of a module over an algebra. (Once again, our definition differs from that of the standard references [11, 17] in that we do not require a coalgebra to have a counit.)

DEFINITION. If C is a coalgebra with comultiplication Δ_C, then a (*left*) *comodule* over C is a vector space M together with a linear map $\Delta_M : M \to C \otimes M$ such that $(\Delta_C \otimes 1)\Delta_M = (1 \otimes \Delta_M)\Delta_M$.

Just as any algebra is a module over itself with multiplication map equal to the multiplication of the algebra, any coalgebra is a comodule over itself with comultiplication equal to the comultiplication of the coalgebra.

EXAMPLE. If C is a coalgebra, then C is a comodule over itself with $\Delta_M = \Delta_C$.

Just as a map of modules over an algebra is defined to be a linear map that commutes with the multiplication map of the module, a map of comodules over a coalgebra is linear map that commutes with the comultiplication map of the comodule.

DEFINITION. If M and N are comodules over the coalgebra C, then a *comodule map* $f\colon M \to N$ is defined to be a linear map $f\colon M \to N$ such that $\Delta_n f = (1 \otimes f)\, \Delta_M$.

EXAMPLE. If the coalgebra C is considered as a comodule over itself, then a comodule map $f\colon C \to C$ is just a linear map $f\colon C \to C$ such that $\Delta_C f = (1 \otimes f)\, \Delta_C$.

Theorem 6.13 now implies the following.

THEOREM 6.21. *The algebra of descending operators is exactly the algebra of comodule maps* $P \to P$.

7. NEWTONIAN SEQUENCES

7.1. *Classifying Newtonian Sequences*

Given a Newtonian sequence $\{p_n\}$, the constant terms $(\varepsilon(p_0), \varepsilon(p_1), \varepsilon(p_2), \ldots)$ form a sequence in K with $\varepsilon(p_0) = p_0 \neq 0$. In this section, we show that this sequence entirely determines the Newtonian sequence, and that any sequence in K with first term non-zero arises as the sequence of constant terms of a Newtonian sequence. We also show how to construct the Newtonian sequence having a given sequence as its sequence of constant terms.

THEOREM 7.1. *There is a one-to-one correspondence between the set of Newtonian sequences and the set of sequences* $\{(a_0, a_1, a_2, \ldots) \mid a_i \in K, a_0 \neq 0\}$ *defined by taking the constant terms in the Newtonian sequence.*

Proof. Let $(a_0, a_1, a_2, \ldots)$ be a sequence in K such that $a_0 \neq 0$; we will show that there is a unique Newtonian sequence having the a_i as its constant terms. We will do this by proving inductively that for each $n \geqslant 0$ there is a unique set of polynomials $\{p_0, p_1, p_2, \ldots, p_n\}$ such that both $\varepsilon(p_k) = a_k$ and $\Delta p_k = \sum_{i=0}^{k-1} p_i \otimes p_{k-1-i}$ for $0 \leqslant k \leqslant n$.

The induction is begun by letting $p_0 = a_0$. We now assume that there is a unique set of polynomials p_0 through p_n satisfying the conditions of the last paragraph; we must show that there is a unique way to define p_{n+1} such that $\varepsilon(p_{n+1}) = a_{n+1}$ and $\Delta(p_{n+1}) = \sum_{i=0}^{n} p_i \otimes p_{n-i}$.

If $q = b_{n+1}x^{n+1} + b_n x^n + \cdots + b_0$, then $\Delta(q) = b_{n+1}$ (terms of total degree n) $+ b_n$ (terms of total degree $n-1$) $+ \cdots + b_1(1 \otimes 1)$. Thus, q will satisfy $\Delta(q) = \sum_{i=0}^{n} p_i \otimes p_{n-i}$ if and only if, in the expression $\sum_{i=0}^{n} p_i \otimes p_{n-i}$, each of the terms $x^n \otimes 1, x^{n-1} \otimes x, ..., 1 \otimes x^n$ has the coefficient b_{n+1}, each of the terms $x^{n-1} \otimes 1, ..., 1 \otimes x^{n-1}$ has the coefficient b_n, etc. This shows two things. The first is that there is at most one $n+1$st degree polynomial q for which both $\Delta(q) = \sum_{i=0}^{n} p_i \otimes p_{n-i}$ and $\varepsilon(q) = a_{n+1}$. The second is that to show that there does exist such a polynomial, it is sufficient to show that there is *some* Newtonian sequence $\{q_k\}$ for which $q_k = p_k$ for $k \leqslant n$. To complete the proof, we will show that there is such a Newtonian sequence.

We will obtain the Newtonian sequence that we need by constructing a basic functional f and then taking its associated Newtonian sequence. Since $\{p_0, p_1, ..., p_n\}$ is a basis of the space of polynomials of degree less than or equal to n, we can complete it to a basis of all of P by adding the polynomials $\{x^{n+1}, x^{n+2}, ...\}$. We can now define the functional f by letting $f(p_0) = 1$, $f(p_k) = 0$ for $1 \leqslant k \leqslant n$, and letting $f(x^k) = 0$ for $k > n$. Since $p_0 \neq 0$, $f(1) = (1/p_0) f(p_0) = 1/p_0 \neq 0$, and so f is a basic functional. Let $\{q_k\}$ be the Newtonian sequence associated with f. To complete the proof, we need only show that $q_k = p_k$ for $k \leqslant n$. To do this, we need to show that $f^{k+1}(p_m) = \delta_{m,k}$ for $m \leqslant n$. We will do this with an induction on k.

To begin the induction, we note that f was defined so that $f(p_0) = 1$ and $f(p_m) = 0$ for $1 \leqslant m \leqslant n$. The induction step follows by noting that if $m < k$ Corollary 3.8 implies that $f^{k+1}(p_m) = 0$, if $m = k$ then we have $f^{k+1}(p_k) = (f \otimes f^k)\, \Delta(p_k) = \sum_{i=0}^{k-1} f(p_i)\, f^k(p_{k-1-i})$ the only non-zero term of which is $f(p_0)\, f^k(p_{k-1}) = 1$, and if $m > k$ then $f^{k+1}(p_m) = \sum_{i=0}^{m-1} f(p_i)\, f^k(p_{m-1-i})$, all of whose terms are zero. This completes the proof of the theorem.

We will now show how to inductively construct the Newtonian sequence having a given sequence as its sequence of constant terms. We will need the following lemma.

LEMMA 7.2. *If $p \in P$, then*

$$p = \varepsilon(p) + x\varepsilon^*(p).$$

Proof. It is straightforward to verify this for the basis $\{x^n\}$ of P.

PROPOSITION 7.3. *If $(a_0, a_1, a_2, ...)$ is a sequence in K with $a_0 \neq 0$, then* *we can inductively construct the Newtonian sequence with $(a_0, a_1, a_2, \ldots)$ as its sequence of constant terms by letting*

$$p_n = a_n + x \sum_{i=0}^{n-1} a_i p_{n-1-i}.$$

Proof. Theorem 7.1 implies that there exists a Newtonian sequence $\{p_n\}$ for which $\varepsilon(p_n) = a_n$ for all $n \geqslant 0$, Since $\varepsilon^* = (\varepsilon \otimes 1)\Delta$, Lemma 7.2 implies that

$$\begin{aligned} p_n &= \varepsilon(p_n) + x\varepsilon^*(p_n) \\ &= a_n + x(\varepsilon \otimes 1) \sum_{i=0}^{n-1} p_i \otimes p_{n-1-i} \\ &= a_n + x \sum_{i=0}^{n-1} a_i p_{n-1-i}, \end{aligned}$$

and the proof is complete.

7.2. *The Generating Function of a Newtonian Sequence*

In this section, we characterize the generating function of a Newtonian sequence.

DEFINITIONS. The *generating function* of the polynomial sequence $\{p_n\}$ is the formal power series

$$\sum_{k=0}^{\infty} p_k \varepsilon^{k+1} = p_0 \varepsilon + p_1 \varepsilon^2 + p_2 \varepsilon^3 + \cdots.$$

This is an element of the augmentation ideal of $P[[\varepsilon]]$, the algebra of formal power series in the variable ε with coefficients in the algebra $P = K[x]$. We will call this augmentation ideal the *space of generating functions.*

Thus, the space of generating functions is the vector space of all formal power series $\sum_{k=0}^{\infty} p_k(x)\, \varepsilon^{k+1}$, where the $p_k(x)$ can be of any degree. The series $\sum_{k=0}^{\infty} p_k(x)\, \varepsilon^{k+1}$ is the generating function of a polynomial sequence if and if each $p_k(x)$ is of degree k, and the following is easy to verify.

PROPOSITION 7.4. *If $f \in P^*$ and $\{p_k(x)\}$ is a sequence in P, then $\sum_{k=0}^{\infty} p_k(x) f^{k+1}$ is a well-defined element of the space of generating functions.*

A generating function defines a linear operator on P, where the term $p\varepsilon^{n+1}$ denotes the operator that sends x^n to p and sends x^k for $k \neq n$ to zero. Thus, the generating function of the polynomial sequence $\{p_n\}$ yields the operator that sends x^n to p_n, and it is easy to verify the following.

PROPOSITION 7.5. *Two generating functions are equal if and only if they define the same operator on P.*

We can now characterize the generating function of a Newtonian sequence.

THEOREM 7.6. *The sequence $\{p_n(x)\}$ is a Newtonian sequence if and only if there is a basic functional f such that*

$$\sum_{k=0}^{\infty} p_k(x)\,\varepsilon^{k+1} = \frac{f}{1-xf} = f + xf^2 + x^2f^3 + \cdots.$$

Proof. If $\{p_n(x)\}$ is a Newtonian sequence, let g be its associated basic functional. If f is the inverse of g under functional composition, then Proposition 5.9 implies that $\sum_{k=0}^{\infty} p_k(x)\,\varepsilon^{k+1} = \phi_f^*$. If $\{q_n(x)\}$ is the Newtonian sequence associated with f, then $f/(1-xf) = f + xf^2 + x^2f^3 + \cdots$ is the operator that takes $q_n(x)$ to x^n, and so Proposition 5.9 implies that $f/(1-xf) = \phi_f^*$. Thus, Proposition 7.5 implies that $\sum_{k=0}^{\infty} p_k(x)\,\varepsilon^{k+1} = f/(1-xf)$.

Conversely, assume that f is a basic functional such that $\sum_{k=0}^{\infty} p_k(x)\,\varepsilon^{k+1} = f/(1-xf)$. If $\{q_n(x)\}$ is the Newtonian sequence associated with f, then $f/(1-xf)$ is the operator that takes $q_n(x)$ to x^n, and so Proposition 5.9 implies that $f/(1-xf) = \phi_f^*$, which is a coalgebra automorphism of P. Since $\sum_{k=0}^{\infty} p_k(x)\,\varepsilon^{k+1}$ takes x^n to $p_n(x)$, Theorem 5.6 implies that $\{p_n(x)\}$ is a Newtonian sequence, and the proof is complete.

The proof of Theorem 7.6 also shows the following.

COROLLARY 7.7. *If $\{p_n(x)\}$ is a Newtonian sequence with associated basic functional f, then*

$$\sum_{k=0}^{\infty} p_k(x)\,\varepsilon^{k+1} = \frac{\bar{f}}{1-x\bar{f}}$$

(where $\bar{f}$ is the inverse of f under functional composition).

7.3. *The Newtonian Sequence Associated with a Basic Funtional*

In this section, we give several constructions of the Newtonian sequence associated with a basic functional f.

THEOREM 7.8. *Let f be a basic functional. If we factor f in $K[[\varepsilon]]$ as $f = \varepsilon M$, then its associated Newtonian sequence $\{p_n\}$ is given by*

$$p_n = \frac{1}{n+1} D(x(M^{-(n+1)})^* x^n).$$

The proof of Theorem 7.8 will use the following lemma.

LEMMA 7.9. *If $f \in P^*$ and $p \in P$, then*

$$\langle f \mid D(xp) \rangle = \langle \varepsilon Df \mid p \rangle$$

(*where Df means differentiation with respect to ε*).

Proof. It is sufficient to consider the case $f = \varepsilon^k$ and $p = x^n$. In this case,

$$\begin{aligned} \langle f \mid D(xp) \rangle &= \langle \varepsilon^k \mid (n+1)\, x^n \rangle \\ &= \begin{cases} n+1, & \text{if } k = n+1 \\ 0, & \text{if } k \neq n+1 \end{cases} \end{aligned}$$

while

$$\begin{aligned} \langle \varepsilon Df \mid p \rangle &= \langle k\varepsilon^k \mid x^n \rangle \\ &= \begin{cases} k, & \text{if } k = n+1 \\ 0, & \text{if } k \neq n+1. \end{cases} \end{aligned}$$

This completes the proof.

Proof of Theorem 7.8. We must show that $f^{k+1}(p_n) = \delta_{n,k}$. If $k > n$, then Corollary 3.8 implies that $f^{k+1}(p_n) = 0$. If $k = n$, then, if $M = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$, we have

$$\begin{aligned} \langle f^{k+1} \mid p_n \rangle &= \left\langle \varepsilon^{k+1} M^{k+1} \,\middle|\, \frac{1}{n+1} D(x(a_0^{-n-1} x^n + \text{lower degree terms})) \right\rangle \\ &= \langle \varepsilon^{n+1} M^{n+1} \mid a_0^{-n-1} x^n + \text{lower degree terms} \rangle \\ &= \langle \varepsilon^{n+1} \mid (M^{n+1})^* (a_0^{-n-1} x^n + \text{lower degree terms} \rangle \\ &= \langle \varepsilon^{n+1} \mid x^n + \text{lower degree terms} \rangle \end{aligned}$$

If $k<n$, then

$$\begin{aligned}
&\langle f^{k+1} \mid p_n \rangle \\
&= \left\langle f^{k+1} \,\middle|\, \frac{1}{n+1} D(x(M^{-(n+1)})^* x^n) \right\rangle \\
&= \frac{1}{n+1} \langle \varepsilon D(f^{k+1}) \mid (M^{-(n+1)})^* x^n \rangle \\
&= \frac{1}{n+1} \langle \varepsilon D(\varepsilon^{k+1} M^{k+1}) \mid (M^{-(n+1)})^* x^n \rangle \\
&= \frac{1}{n+1} \langle (k+1)\, \varepsilon^{k+2} M^k M' + (k+1)\, \varepsilon^{k+1} M^{k+1} \mid (M^{-(n+1)})^* x^n \rangle \\
&= \frac{k+1}{n+1} \langle \varepsilon^{k+2} M^k M' \mid (M^{-(n+1)})^* x^n \rangle \\
&\quad + \frac{k+1}{n+1} \langle \varepsilon^{k+1} M^{k+1} \mid (M^{-(n+1)})^* x^n \rangle \\
&= \frac{k+1}{n+1} \langle \varepsilon^{k+2} M^{k-n-1} M' \mid x^n \rangle + \frac{k+1}{n+1} \langle \varepsilon^{k+1} M^{k-n} \mid x^n \rangle \\
&= \frac{k+1}{n+1} \langle \varepsilon M^{k-n-1} M' \mid x^{n-k-1} \rangle + \frac{k+1}{n+1} \langle \varepsilon M^{k-n} \mid x^{n-k} \rangle \\
&= \frac{k+1}{n+1} \left\langle \frac{1}{k-n} \varepsilon (M^{k-n})' \,\middle|\, x^{n-k-1} \right\rangle + \frac{k+1}{n+1} \langle \varepsilon M^{k-n} \mid x^{n-k} \rangle.
\end{aligned}$$

If $M^{k-n} = b_0 + b_1 \varepsilon + b_2 \varepsilon^2 + \cdots$, then the first term equals

$$\frac{k+1}{n+1} \left\langle \frac{n-k}{k-n} b_{n-k} \varepsilon^{n-k} \,\middle|\, x^{n-k-1} \right\rangle = -\frac{k+1}{n+1} b_{n-k},$$

while the second term equals

$$\frac{k+1}{n+1} \langle b_{n-k} \varepsilon^{n-k+1} \mid x^{n-k} \rangle = \frac{k+1}{n+1} b_{n-k}.$$

Thus, the sum is zero, and the proof is complete.

The next theorem gives an inductive construction of the Newtonian sequence associated with a basic functional.

THEOREM 7.10. *Let f be a basic functional, with associated Newtonian sequence $\{p_n\}$. If we factor f in $K[[\varepsilon]]$ as $f=\varepsilon M$, then for all $n \geqslant 0$,*

$$p_{n+1} = (M^{-1})^*(xp_n).$$

Proof. We showed in Example 6.2 that if $n \geqslant 0$, then $f^*(p_{n+1}) = p_n$. Suppose we knew that $xf^*(p_{n+1}) = f^*(xp_{n+1})$; in this case, we would have

$$\begin{aligned}(M^{-1})^*(xp_n) &= (M^{-1})^*(xf^*p_{n+1}) \\ &= (M^{-1})^* f^*(xp_{n+1}) \\ &= (M^{-1})^* M^*\varepsilon^*(xp_{n+1}) \\ &= \varepsilon^*(xp_{n+1}) \\ &= p_{n+1}.\end{aligned}$$

It remains only to show that $xf^*(p_{n+1}) = f^*(xp_{n+1})$. Let $f = a_0\varepsilon + a_1\varepsilon^2 + a_2\varepsilon^3 + \cdots$ and let $p_{n+1} = b_{n+1}x^{n+1} + b_nx^n + \cdots + b_0$; then

$$\begin{aligned}f^*(xp_{n+1}) - xf^*(p_{n+1}) &= a_0b_0 + a_1b_1 + a_2b_2 + \cdots + a_{n+1}b_{n+1} \\ &= \langle f \mid p_{n+1}\rangle \\ &= 0,\end{aligned}$$

since $n+1>0$. This completes the proof.

If f is a basic functional, then Theorem 5.1 gives us a continuous automorphism ϕ_f of P^* such that $\phi_f(\varepsilon) = f$.

Notation. If f is a basic functional, we let $\bar{f}$ denote the basic functional such that $\phi_{\bar{f}} = \phi_f^{-1}$. This is equivalent to saying that $\bar{f}$ is the unique element of P^* such that $\phi_f(\bar{f}) = \varepsilon$.

The functional $\bar{f}$ is the inverse of the basic functional f in the group of continuous automorphisms of P^* (where we are using Theorem 5.1 to identify the basic functional f with the continuous automorphism of P^* that takes ε to f). It we use Theorem 3.7 to think of elements of P^* as formal power series in ε with zero constant term, then the group operation is *functional composition*: if $f = a_0\varepsilon + a_1\varepsilon^2 + a_2\varepsilon^3 + \cdots$ and $g = b_0\varepsilon + b_1\varepsilon^2 + b_2\varepsilon^3 + \cdots$, then $fg = a_0 g + a_1 g^2 + a_2 g^3 + \cdots$, and the identity element of this group is $\mathrm{Id} = \varepsilon$. The next theorem shows how you can construct the Newtonian sequence associated with the basic functional f from knowledge of $\bar{f}$.

THEOREM 7.11. *If f is a basic functional, then its associated Newtonian sequence $\{p_n\}$ is given by*

$$p_n = \langle \bar{f}^{n+1} | x^n \rangle x^n + \langle \bar{f}^n | x^n \rangle x^{n-1} + \langle \bar{f}^{n-1} | x^n \rangle x^{n-2} + \cdots + \langle \bar{f} | x^n \rangle.$$

Proof. Let $p_n = \sum_{k=0}^{n} c_{nk} x^k$; we must show that $c_{nk} = \langle \bar{f}^{k+1} | x^n \rangle$. We proved in Proposition 5.9 that $p_n = \phi_{\bar{f}}^*(x^n)$ (where $\phi_{\bar{f}}$ is the continuous automorphism of P^* that takes ε to $\bar{f}$). Thus,

$$\begin{aligned} c_{nk} &= \langle \varepsilon^{k+1} | p_n \rangle \\ &= \langle \varepsilon^{k+1} | \phi_{\bar{f}}^*(x^n) \rangle \\ &= \langle \phi_{\bar{f}}(\varepsilon^{k+1}) | x^n \rangle \\ &= \langle \bar{f}^{k+1} | x^n \rangle. \end{aligned}$$

We proved in Theorem 7.1 that a Newtonian sequence $\{p_n\}$ is entirely determined by its sequence of constant terms $(\varepsilon(p_0), \varepsilon(p_1), \varepsilon(p_2), \ldots)$, and that this can be chosen arbitrarily as long as $\varepsilon(p_0) \neq 0$. We also proved (in Theorem 5.7) that a Newtonian sequence is classified by its associated basic functional $f = a_0 \varepsilon + a_1 \varepsilon^2 + a_2 \varepsilon^3 + \cdots$, and that this can be chosen arbitrarily as long as $a_0 \neq 0$. Theorem 7.11 allows us to relate these two classifications of Newtonian sequences.

COROLLARY 7.12. *If f is a basic functional with associated Newtonian sequence $\{p_n\}$, then the constant term of p_n equals the coefficient of ε^{n+1} in $\bar{f}$, the inverse to f under functional composition.*

We will show in the next section that these ideas provide an easy proof of the Lagrange inversion formula.

7.4. *The Lagrange Inversion Formula*

In this section, we use the ideas of the last section to obtain an easy proof of the Lagrange inversion formula.

THEOREM (The Lagrange Inversion Formula) 7.13. *Let $f(t) \in K[[t]]$ have constant term equal to zero and non-zero first degree term (so that $\bar{f}(t)$, the inverse to $f(t)$ under functional composition, exists). If $g(t) \in K[[t]]$, then the coefficient of t^n in $g(\bar{f}(t))$ equals $1/n$ times the coefficient of t^{n-1} in $(f(t)/t)^{-n} g'(t)$.*

Proof. Factor f in $K[[\varepsilon]]$ as $f = \varepsilon M$. We will show that

$$\langle g(\bar{f}(\varepsilon)) | x^{n-1} \rangle = \left\langle \frac{\varepsilon}{n} M^{-n} g'(\varepsilon) \,\middle|\, x^{n-1} \right\rangle.$$

We have

$$\begin{aligned}
\langle g(\bar{f}(\varepsilon)) \mid x^{n-1}\rangle &= \langle \phi_f(g(\varepsilon)) \mid x^{n-1}\rangle \\
&= \langle g(\varepsilon) \mid \phi_f^* x^{n-1}\rangle \\
&= \langle g(\varepsilon) \mid p_{n-1}\rangle \qquad \text{(using Proposition 5.9)} \\
&= \left\langle g(\varepsilon) \,\middle|\, \frac{1}{n} D(x(M^{-n})^* x^{n-1})\right\rangle \qquad \text{(using Theorem 7.8)} \\
&= \left\langle \frac{\varepsilon}{n} g'(\varepsilon) \,\middle|\, (M^{-n})^* x^{n-1}\right\rangle \qquad \text{(using Lemma 7.9)} \\
&= \left\langle \frac{\varepsilon}{n} M^{-n} g'(\varepsilon) \mid x^{n-1}\right\rangle,
\end{aligned}$$

and the proof is complete.

For an explanation of how Theorem 7.13 implies the other standard forms of the Lagrange Inversion Formula, see [5, Sect. 3.8].

If we take $g(t) = t$ in Theorem 7.13, we obtain a formula for $\bar{f}(t)$, the inverse to $f(t)$ under composition.

COROLLARY 7.14. *Let $f(t) \in K[[t]]$ have constant term zero and non-zero first degree term. If $\bar{f}(t) = a_1 t + a_2 t^2 + a_3 t^3 + \cdots$ is the inverse of $f(t)$ under composition, then*

$$a_n = \frac{1}{n} \text{ (the coefficient of } t^{n-1} \text{ in } (f(t)/t)^{-n}).$$

7.5. *Examples*

In this section, we give a number of examples of Newtonian sequences.

EXAMPLE 7.15. If $a, b \in K$ and $a \neq 0$, then

$$p_n = \frac{1}{a^{n+1}} (x-b)^n$$

is the Newtonian sequence associated with the basic functional $f = a\varepsilon + ab\varepsilon^2 + ab^2\varepsilon^3 + \cdots$.

Proof. We will use Theorem 7.10. If we factor f as $f = \varepsilon M$, then

$M = a + ab\varepsilon + ab^2\varepsilon^2 + \cdots = a/(1 - b\varepsilon)$. Thus, $M^{-1} = (1 - b\varepsilon)/a$, and so $p_{n+1} = (M^{-1})^*(xp_n) = (x - b)\, p_n/a$. Since $p_0 = (1/f(1)) = 1/a$, the proof is complete.

EXAMPLE 7.16. *The polynomial sequence*

$$p_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{2n+2}{n-k} \frac{k+1}{n+1} x^k$$

is the Newtonian sequence associated with the basic functional $f = \varepsilon + 2\varepsilon^2 + 3\varepsilon^3 + \cdots$.

Proof. We will use Theorem 7.8. If we factor f as $f = \varepsilon M$, then $M = 1 + 2\varepsilon + 3\varepsilon^2 + \cdots = 1/(1-\varepsilon)^2$. Thus, $M^{-1} = (1-\varepsilon)^2$, and so

$$\begin{aligned} M^{-(n+1)} &= (1-\varepsilon)^{2(n+1)} \\ &= \sum_{k=0}^{2n+2} (-1)^k \binom{2n+2}{k} \varepsilon^k. \end{aligned}$$

Thus,

$$\begin{aligned} p_n &= \frac{1}{n+1} D\left(x \sum_{k=0}^{2n+2} (-1)^k \binom{2n+2}{k} (\varepsilon^*)^k x^n\right) \\ &= \frac{1}{n+1} D\left(x \sum_{k=0}^{n} (-1)^k \binom{2n+2}{k} x^{n-k}\right) \\ &= \frac{1}{n+1} D \sum_{k=0}^{n} (-1)^k \binom{2n+2}{k} x^{n+1-k} \\ &= \sum_{k=0}^{n} (-1)^k \binom{2n+2}{k} \frac{n+1-k}{n+1} x^{n-k} \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{2n+2}{n-k} \frac{k+1}{n+1} x^k. \end{aligned}$$

EXAMPLE 7.17. If $a \in K$, then

$$p_n = \sum_{k=0}^{n} (-1)^{n-k} \frac{a^{n-k}(n+1)^{n-k-1}(k+1)}{(n-k)!} x^k$$

is the Newtonian sequence associated with the basic functional $f = \varepsilon + a\varepsilon^2 + (a^2/2!)\, \varepsilon^3 + (a^3/3!)\, \varepsilon^4 + \cdots$.

Proof. We will use Theorem 7.8. If we factor f as $f=\varepsilon M$, then $M=1+a\varepsilon+(a^2/2!)\,\varepsilon^2+(a^3/3!)\,\varepsilon^3+\cdots=e^{a\varepsilon}$. Thus, $M^{-1}=e^{-a\varepsilon}$, and so

$$\begin{aligned}
M^{-(n+1)}&=(e^{a\varepsilon})^{-(n+1)}\\
&=e^{-a(n+1)\varepsilon}\\
&=1-a(n+1)\varepsilon+\frac{a^2(n+1)^2}{2!}\varepsilon^2-\frac{a^3(n+1)^3}{3!}\varepsilon^3+\cdots\\
&=\sum_{k=0}^{\infty}(-1)^k\frac{a^k(n+1)^k}{k!}\varepsilon^k.
\end{aligned}$$

Thus,

$$\begin{aligned}
p_n&=\frac{1}{n+1}D\left(x\sum_{k=0}^{\infty}(-1)^k\frac{a^k(n+1)^k}{k!}(\varepsilon^*)^k\,x^n\right)\\
&=\frac{1}{n+1}D\left(x\sum_{k=0}^{n}(-1)^k\frac{a^k(n+1)^k}{k!}x^{n-k}\right)\\
&=\frac{1}{n+1}D\sum_{k=0}^{n}(-1)^k\frac{a^k(n+1)^k}{k!}x^{n+1-k}\\
&=\sum_{k=0}^{n}(-1)^k\frac{a^k(n+1)^k}{k!}\,\frac{n+1-k}{n+1}x^{n-k}\\
&=\sum_{k=0}^{n}(-1)^k\frac{a^k(n+1)^{k-1}(n+1-k)}{k!}x^{n-k}\\
&=\sum_{k=0}^{n}(-1)^{n-k}\frac{a^{n-k}(n+1)^{n-k-1}(k+1)}{(n-k)!}x^k.
\end{aligned}$$

EXAMPLE 7.18. If $a\in K$, then

$$p_n=\sum_{k=0}^{[n/2]}(-1)^k\frac{a^k(n+1)^{k-1}(n+1-2k)}{k!}x^{n-2k}$$

is the Newtonian sequence associated with the basic functional $f=\varepsilon+a\varepsilon^2+(a^2/2!)\,\varepsilon^4+(a^3/3!)\,\varepsilon^6+\cdots$.

Proof. We will use Theorem 7.8. If we factor f as $f=\varepsilon M$, then $M=1+a\varepsilon^2+(a^2/2!)\,\varepsilon^4+(a^3/3!)\,\varepsilon^6+\cdots=e^{a\varepsilon^2}$. Thus,

$$M^{-(n+1)} = e^{-a(n+1)\varepsilon^2}$$

$$= 1 - a(n+1)\,\varepsilon^2 + \frac{a^2(n+1)^2}{2!}\varepsilon^4 - \frac{a^3(n+1)^3}{3!}\varepsilon^6 + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{a^k(n+1)^k}{k!} \varepsilon^{2k}$$

Thus,

$$p_n = \frac{1}{n+1} D\left(x \sum_{k=0}^{\infty} (-1)^k \frac{a^k(n+1)^k}{k!} (\varepsilon^*)^{2k} x^n\right)$$

$$= \frac{1}{n+1} D\left(x \sum_{k=0}^{[n/2]} (-1)^k \frac{a^k(n+1)^k}{k!} x^{n-2k}\right)$$

$$= \frac{1}{n+1} D \sum_{k=0}^{[n/2]} (-1)^k \frac{a^k(n+1)^k}{k!} x^{n+1-2k}$$

$$= \sum_{k=0}^{[n/2]} (-1)^k \frac{a^k(n+1)^k}{k!} \frac{n+1-2k}{n+1} x^{n-2k}$$

$$= \sum_{k=0}^{[n/2]} (-1)^k \frac{a^k(n+1)^{k-1}(n+1-2k)}{k!} x^{n-2k}.$$

Example 7.19. The polynomial sequence

$$p_n = \sum_{k=0}^{n} \binom{2n-k}{n} \frac{k+1}{n+1} x^k$$

is the Newtonian sequence associated with the basic functional $f = \varepsilon - \varepsilon^2$.

Proof. We will use Theorem 7.8. If we factor f as $f = \varepsilon M$, then $M = 1 - \varepsilon$. Thus, $M^{-1} = 1/(1-\varepsilon)$, and so $D^n(M^{-1}) = n!M^{-(n+1)}$. We can now write

$$M^{-(n+1)} = \frac{1}{n!} D^n(M^{-1})$$

$$= \frac{1}{n!} D^n(1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \cdots)$$

$$= \binom{n}{0} + \binom{n+1}{n}\varepsilon + \binom{n+2}{n}\varepsilon^2 + \binom{n+3}{n}\varepsilon^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \binom{n+k}{n} \varepsilon^k.$$

Thus,

$$\begin{aligned}
p_n &= \frac{1}{n+1} D\left(x \sum_{k=0}^{\infty} \binom{n+k}{n} (\varepsilon^*)^k x^n\right) \\
&= \frac{1}{n+1} D\left(x \sum_{k=0}^{n} \binom{n+k}{n} x^{n-k}\right) \\
&= \frac{1}{n+1} D \sum_{k=0}^{n} \binom{n+k}{n} x^{n+1-k} \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+k}{n} (n+1-k)\, x^{n-k} \\
&= \sum_{k=0}^{n} \binom{2n-k}{n} \frac{k+1}{n+1} x^k.
\end{aligned}$$

8. Scheffer Sets

8.1. *Definition and Classification*

We showed in Example 6.2 that if f is a basic functional with associated Newtonian sequence $\{p_n\}$, then $f^*(p_n) = p_{n-1}$ for $n > 0$. We show in this section that there are many other polynomial sequences for which this is also true.

Definition. If f is a basic functional, then the polynomial sequence $\{s_n\}$ will be called a *Sheffer set for f* (or a *Scheffer sequence* for f) if $f^*(s_n) = s_{n-1}$ for $n > 0$. A polynomial sequence will be called a *Sheffer set* (or a *Sheffer sequence*) if it is a Sheffer set for some basic functional.

We will show that a given basic functional f has one Sheffer set for every invertible descending operator.

Proposition 8.1. *Let f be a basic functional with associated Newtonian sequence $\{p_n\}$. The polynomial sequence $\{s_n\}$ is a Sheffer set for f if and only if there is an invertible descending operator M^* such that $s_n = M^*(p_n)$ for $n \geqslant 0$.*

Proof. Let M^* be an invertible descending operator, and let $s_n = M^*(p_n)$. Since M^* is invertible, Proposition 6.11 implies that s_n is of degree n, and we have

$$\begin{aligned} f^*(s_n) &= f^*M^*(p_n) \\ &= M^*f^*(p_n) \\ &= M^*(p_{n-1}) \\ &= s_{n-1} \end{aligned}$$

for $n > 0$.

Conversely, let $\{s_n\}$ be a Sheffer set for f. Since $\{p_n\}$ and $\{s_n\}$ are bases of P, we can define an invertible linear operator L on P by letting $L(p_n) = s_n$. To show that L is a descending operator, it suffices (by Theorem 6.13) to show that $f^*L = Lf^*$. Since

$$\begin{aligned} f^*L(p_n) &= f^*s_n \\ &= s_{n-1} \\ &= L(p_{n-1}) \\ &= Lf^*(p_n) \end{aligned}$$

and $\{p_n\}$ is a basis of P, the proof is complete.

Since $\{p_n\}$ is a basis of P, two operators that agree on $\{p_n\}$ must be equal. Thus, we have proved the following.

COROLLARY 8.2. *If f is a basic functional with associated Newtonian sequence $\{p_n\}$, then there is a one-to-one correspondence between the set of Sheffer sets for f and the set of invertible descending operators defined by taking the invertible descending operator M^* to $\{M^*(p_n)\}$.*

There is an obvious parallel between the classification of Newtonian sequences and the classification of Sheffer sets of a given basic functional f: Each Newtonian sequence corresponds to a basic functional and each Sheffer set for f corresponds to an invertible descending operator. (Each invertible descending operator corresponds to a basic functional by taking the invertible descending operator M^* (for $M = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$ with $a_0 \neq 0$) to the basic functional $a_0\varepsilon + a_1\varepsilon^2 + a_2\varepsilon^3 + \cdots$.) We will now show that this parallelism extends further: Just as we proved in Theorem 7.1 that a Newtonian sequence is determined by its sequence of constant terms (and that this sequence can be chosen arbitrarily, as long as the first term is non-zero), we will now show that a Sheffer set for a basic functional f is determined by its constant terms, and that these can be chosen arbitrarily (as long as the first term is non-zero).

THEOREM 8.3. *Let f be a basic functional. There is a one-to-one correspondence between the set of Sheffer sets for f and the set of sequences $\{(a_0, a_1, a_2, \ldots) \mid a_i \in K, a_0 \neq 0\}$ defined by taking the constant terms in the Sheffer set.*

Proof. Let $\{p_n\}$ be the Newtonian sequence associated with f. We proved in Corollary 8.2 that there is a one-to-one correspondence between the set of Sheffer sets for f and the set of invertible descending operators, defined by sending the invertible descending operator M^* to the Sheffer set $\{M^*(p_n)\}$. We will show here that for each sequence $(a_0, a_1, a_2, \ldots)$ in K with $a_0 \neq 0$, there is a unique invertible descending operator M^* such that $\langle \varepsilon \mid M^*(p_n) \rangle = a_n$ for all $n \geqslant 0$.

Corollary 6.18 implies that it is sufficient to show that for each sequence $(a_0, a_1, a_2, \ldots)$ in K with $a_0 \neq 0$, there exists a unique $M \in K[[\varepsilon]]$ such that $\langle \varepsilon \mid M_f^*(p_n) \rangle = a_n$ for all $n \geqslant 0$. We will show by induction on n that there is a unique choice of $b_n \in K$ such that if $M = b_0 + b_1\varepsilon + b_2\varepsilon^2 + \cdots$, then $\langle \varepsilon \mid M_f^*(p_k) \rangle = a_k$ for $k \leqslant n$. To begin the induction, we note that since $p_0 \neq 0$, we must set $b_0 = a_0/p_0$ in order to have $M_f^*(p_0) = b_0 p_0 = a_0$. We now assume that there are unique values of $b_0, b_1, \ldots, b_{n-1}$ and b_n for which $\langle \varepsilon \mid M_f^*(p_k) \rangle = a_k$ for $k \leqslant n$. Since $M_f^*(p_{n+1}) = b_0 p_{n+1} + b_1 p_n + b_2 p_{n-1} + \cdots + b_{n+1} p_0$, we have $\langle \varepsilon \mid M_f^*(p_{n+1}) \rangle = b_0 \varepsilon(p_{n+1}) + b_1 \varepsilon(p_n) + b_2 \varepsilon(p_{n-1}) + \cdots + b_{n+1} \varepsilon(p_0)$. Since b_0, $b_1, \ldots$ and b_n are already determined and $\varepsilon(p_0) = p_0 \neq 0$, there is a unique value of b_{n+1} for which $\langle \varepsilon \mid M_f^*(p_{n+1}) \rangle = a_{n+1}$. This completes the induction step. Since $b_0 \neq 0$, Proposition 6.11 and Corollary 6.19 imply that M_f^* is invertible, and the proof is complete.

8.2. *Comultiplication of Sheffer Sets*

In this section, we give another characterization of Sheffer sets, this time in terms of the comultiplication Δ. We also show how to construct the Sheffer set for a basic functional f having a given sequence as its sequence of constant terms.

THEOREM 8.4. *Let f be a basic functional with associated Newtonian sequence $\{p_n\}$. The polynomial sequence $\{s_n\}$ is a Sheffer set for f if and only if*

$$\Delta(s_n) = \sum_{i=0}^{n-1} s_i \otimes p_{n-1-i}.$$

Proof. Let $\{s_n\}$ be a Scheffer set for f. If M^* is the invertible descending operator such that $M^*(p_n) = s_n$, then we have

$$\begin{aligned}\Delta(s_n) &= \Delta M^*(p_n)\\ &= (M^* \otimes 1)\,\Delta(p_n) \qquad \text{(using Theorem 6.13)}\\ &= (M^* \otimes 1) \sum_{i=0}^{n-1} p_i \otimes p_{n-1-i}\\ &= \sum_{i=0}^{n-1} s_i \otimes p_{n-1-i}.\end{aligned}$$

Conversely, if $\Delta(s_n) = \sum_{i=0}^{n-1} s_i \otimes p_{n-1-i}$, then

$$\begin{aligned}f^*(s_n) &= (\varepsilon_x \otimes f)\,\Delta(s_n)\\ &= (\varepsilon_x \otimes f) \sum_{i=0}^{n-1} s_i \otimes p_{n-1-i}\\ &= s_{n-1},\end{aligned}$$

and the proof is complete.

As an application of this theorem we will show how to construct the Sheffer set for the basic functional f whose constant terms are a given sequence $(a_0, a_1, a_2, ...)$.

PROPOSITION 8.5. *Let f be a basic functional with associated Newtonian sequence $\{p_n\}$, and let $(a_0, a_1, a_2, ...)$ be a sequence in K with $a_0 \neq 0$. If we let*

$$s_n = a_n + x \sum_{i=0}^{n-1} a_i p_{n-1-i},$$

then $\{s_n\}$ is a Scheffer set for f and $\varepsilon(s_n) = a_n$ for all $n \geq 0$.

Proof. Theorem 8.4 implies that there is a Sheffer set $\{s_n\}$ for f such that $\varepsilon(s_n) = a_n$ for all $n \geq 0$; we must show that s_n satisfies the formula of the proposition.

Lemma 7.2 implies that $s_n = \varepsilon(s_n) + x\varepsilon^*(s_n)$, i.e., that $s_n = a_n + x\varepsilon^*(s_n)$. Since $\varepsilon^* = (\varepsilon \otimes 1)\Delta$, Theorem 8.4 implies that

$$\begin{aligned}\varepsilon^*(s_n) &= (\varepsilon \otimes 1) \sum_{i=0}^{n-1} s_i \otimes p_{n-1-i}\\ &= \sum_{i=0}^{n-1} a_i p_{n-1-i},\end{aligned}$$

and the proof is complete.

8.3. *The Associated Functional Sequence of a Sheffer Set*

We show here that just as a Newtonian sequence can be characterized in terms of its associated functional sequence, a Sheffer set can also be characterized in terms of its associated functional sequence. We also obtain an expansion theorem for functionals and operators in terms of their effect on the elements of a Sheffer set.

PROPOSITION 8.6. *Let $(f_1, f_2, f_3, \ldots)$ be a functional sequence with associated polynomial sequence $\{s_n\}$. The polynomial sequence $\{s_n\}$ is a Scheffer set if and only if there is a functional f and an invertible element M of $K[[\varepsilon]]$ such that $f_n = M^{-1}f^n$ for all $n \geqslant 1$.*

Proof. Let $\{s_n\}$ be a Sheffer set for the basic functional f with associated Newtonian sequence $\{p_n\}$, and let $M \in K[[\varepsilon]]$ be the invertible element such that $s_n = M^*(p_n)$ for all $n \geqslant 0$. We then have

$$\begin{aligned}\delta_{n,k} &= \langle f^{n+1} \mid p_k \rangle \\ &= \langle f^{n+1} \mid (M^{-1})^* M^* p_k \rangle \\ &= \langle M^{-1} f^{n+1} \mid M^* p_k \rangle \\ &= \langle M^{-1} f^{n+1} \mid s_k \rangle,\end{aligned}$$

and so we must have $f_n = M^{-1}f^n$ for all $n \geqslant 1$.

Conversely, let f be a functional, and let M be an invertible element of $K[[\varepsilon]]$ such that $f_n = M^{-1}f^n$ for all $n \geqslant 1$. Since

$$\begin{aligned}\delta_{n,k} &= \langle f_{n+1} \mid s_k \rangle \\ &= \langle M^{-1} f^{n+1} \mid s_k \rangle \\ &= \langle f^{n+1} \mid (M^{-1})^* s_k \rangle,\end{aligned}$$

Corollary 3.8 implies that f is a basic functional, and we also have that $\{(M^{-1})^* s_k\}$ must be the Newtonian sequence associated with f. Thus, our result follows from Proposition 8.1, and the proof is complete.

DEFINITIONS. In the situation of Proposition 8.6, we will say that f is the *basic functional associated with the Sheffer set* $\{s_n\}$ and that M^* is the *descending operator associated with the Sheffer set* $\{s_n\}$.

Since two basic functionals whose powers differ by multiplication by a fixed element of $K[[\varepsilon]]$ must be the same, it is easy to see that the basic functional associated with a Sheffer set and the descending operator associated with a Sheffer set are well defined. Theorem 4.3 now implies the following.

THEOREM (Expansion Theorem for Sheffer Sets) 8.7. *Let $\{s_n\}$ be a Sheffer set with associated basic functional f and associated descending operator M^*. If L is either an element pf P^* or a linear operator on P, then*

$$L = \sum_{n=0}^{\infty} L(s_n)\, M^{-1} f^{n+1}.$$

8.4. *The Generating Function of a Sheffer Set*

In this section, we characterize the generating function of a Sheffer set.

THEOREM 8.8. *The sequence $\{s_n(x)\}$ is a Sheffer set if and only if there is a basic functional f and an invertible element M of $K[[\varepsilon]]$ such that*

$$\sum_{k=0}^{\infty} s_k(x)\, \varepsilon^{k+1} = \frac{Mf}{1-xf} = Mf + xMf^2 + x^2 Mf^3 + \cdots.$$

Proof. If $\{s_n(x)\}$ is a Scheffer set, let g be its associated basic functional and let N^* be its associated descending operator. Thus, if $\{p_n(x)\}$ is the Newtonian sequence associated with g and f is the basic functional that is the inverse of g under functional composition, then (using Proposition 5.9)

$$s_n(x) = N^* p_n(x) = N^* \phi_f^* x^n.$$

Since $\sum_{k=0}^{\infty} s_k(x)\, \varepsilon^{k+1}$ takes x^n to $s_n(x)$, we have $\sum_{k=0}^{\infty} s_k(x)\, \varepsilon^{k+1} = N^* \phi_f^*$. Let $M^* = N_f^* = (\phi_f^*)^{-1} N^* \phi_f^*$. Since $\langle Mf^{k+1} | p \rangle = \langle f^{k+1} | M^* p \rangle$ and $f/(1-xf) = f + xf^2 + x^2 f^3 + \cdots = \phi_f^*$ (as in the proof of Theorem 7.6),

$$\begin{aligned} \frac{Mf}{1-xf} &= Mf + xMf^2 + x^2 Mf^3 + \cdots \\ &= \phi_f^* M^* \\ &= \phi_f^* (\phi_f^*)^{-1} N^* \phi_f^* \\ &= N^* \phi_f^*. \end{aligned}$$

Thus, Proposition 7.5 implies that $\sum_{k=0}^{\infty} s_k(x) \varepsilon^{k+1} = Mf/(1-xf)$.

Conversely, assume that we have a basic functional f and an invertible element M of $K[[\varepsilon]]$ such that $\sum_{k=0}^{\infty} s_k(x)\, \varepsilon^{k+1} = Mf/(1-xf)$. Let $N^* = \phi_f^* M^* (\phi_f^*)^{-1}$; then N^* is an invertible descending operator, and (as in the last paragraph) we have $Mf/(1-xf) = \phi_f^* M^* = N^* \phi_f^*$ and ϕ_f^* is a coalgebra automorphism of P. If $p_n(x) = \phi_f^* x^n$, then Theorem 5.6 implies that $\{p_n(x)\}$ is a Newtonian sequence. Thus,

$$\begin{aligned} s_k(x) &= \frac{Mf}{1-xf} x^n \\ &= N^* \phi_f^* x^n \\ &= N^* p_n(x), \end{aligned}$$

and so Proposition 8.1 implies that $\{s_n(x)\}$ is a Scheffer set.

The proof of Theorem 8.8 also implies the following.

COROLLARY 8.9. *If $\{s_n(x)\}$ is a Scheffer set with associated basic functional f and associated descending operator M^*, then*

$$\sum_{k=0}^{\infty} s_k(x)\, \varepsilon^{k+1} = \frac{M_{\bar{f}} \bar{f}}{1 - x\bar{f}}$$

(where $\bar{f}$ is the inverse to f under functional composition, and $M_{\bar{f}}^* = \phi_{\bar{f}}^* M^* \phi_{\bar{f}}^*$).

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